



COUPLED AIR-WATER UNIFORM-FLOW MODEL FOR PARTIALLY FILLED CIRCULAR CONDUIT

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ABSTRACT

Partially filled circular conduits convey water beneath a confined air headspace in many hydraulic systems, including sewers, stormwater networks, and drainage pipes. While the discharge- flow depth relationship of single-phase flow in circular conduits is well established, the role of air motion and air-induced pressure gradients in modifying steady uniform-flow resistance remains poorly understood. Existing air-water flow studies in circular pipes primarily focus on pressurized two-phase transport with imposed phase flow rates, or on transient phenomena such as air pocket compression during rapid filling, leaving a gap in the description of steady, gravity-driven, partially filled flow where air motion is induced rather than prescribed. In such configurations, interfacial shear and headspace confinement can generate longitudinal air pressure gradients that feedback on the water momentum balance and potentially alter the hydraulic capacity.

This study develops a mechanistic, steady uniform-flow model for air-water coupling in partially filled circular conduits. Separate one-dimensional momentum balances are formulated for the water and air phases, explicitly accounting for wall friction, air-water interfacial shear, and the pressure gradient required to sustain air motion in a confined headspace. By eliminating the air pressure gradient through the air momentum equation and substituting it into the water balance, a fully coupled uniform-flow equation is derived. The formulation retains exact circular geometry and naturally reorganizes into a generalized hydraulic resistance kernel expressed in discharge form. This kernel extends the classical Darcy-Weisbach framework by incorporating interfacial momentum exchange and air-pressure-mediated feedback, without relying on empirical correction factors. An air-activation closure and an air-induction parameter are introduced to represent the onset and intensity of air motion in a physically consistent manner.

The resulting model yields an explicit, dimensionless correction function that quantifies the relative contribution of air dynamics to hydraulic resistance as a function of filling rate. Analytical and numerical results demonstrate that air effects are generally weak but finite, localized near high filling rates, strongly dependent on headspace confinement, and vanish smoothly as the conduit approaches full flow. Depending on the intensity of air induction, the air-water interaction function may exhibit a clear maximum or decay monotonically, illustrating how geometry and induced air motion jointly govern coupling behavior.

The novelty of this work lies in providing the first geometry-explicit, steady uniform-flow formulation that rigorously couples air and water momentum balances in a partially filled circular conduit and expresses the result as a generalized hydraulic resistance kernel in discharge form. Unlike existing two-phase pipe-flow models or open-channel formulations, the present approach captures air-pressure feedback induced by confinement under gravity-driven conditions and clarifies how a low-density air phase can influence hydraulic resistance near closure. This framework bridges the gap between classical open-channel hydraulics and two-phase pipeline theory and offers a new mechanistic basis for assessing air effects in partially filled circular conduits.

Keywords: Partially filled circular conduit; Air-water coupling; Uniform flow; Two-phase momentum balance; Interfacial shear; Confined air headspace; Hydraulic resistance kernel; Discharge-flow depth relationship; Sewer and stormwater hydraulics.

INTRODUCTION

Two-phase air-water flow is a long-established and still-active research area because the interaction between phases fundamentally alters momentum transfer, resistance, stability, and energy dissipation. Early theoretical foundations emphasized that gas-liquid flows cannot be reduced to single-phase analogies with simple correction factors; instead, averaged conservation laws must explicitly represent slip, interfacial momentum transfer, and closure relations (Wallis, 1969; Ishii and Hibiki, 2011). These ideas shaped the development of classical two-phase pipe-flow models, particularly for circular conduits, where confinement strongly affects interface geometry and shear distribution. One of the earliest practical approaches for engineering design was the two-phase multiplier concept, which related two-phase pressure losses to single-phase reference flows (Lockhart and Martinelli, 1949). This framework was later given theoretical support by viewing the phases as sharing the conduit and interacting through shear, leading to improved formulations that clarified the role of phase distribution and frictional contributions (Chisholm, 1967).

Air-water interaction is a fundamental feature of many hydraulic structures in which high velocities, strong turbulence, or abrupt changes in flow regime occur. One of the most classical and extensively studied examples is the hydraulic jump, where a supercritical flow transitions abruptly to subcritical conditions (Benabdesselam et al., 2017; Amara et al., 2019; Achour et al., 2022a; 2022b; Achour and Amara, 2023; Bouriche et al., 2023). During this process, intense turbulence and large velocity fluctuations destabilize the free

surface and induce substantial air entrainment into the water body. Large quantities of atmospheric air are entrained through surface roller formation, breaking waves, and plunging jets, resulting in a highly aerated two-phase mixture characterized by strong momentum exchange between phases, energy dissipation, and significant modification of bulk flow properties (Chanson, 1997; Chanson, 2009). The entrained air alters effective density, pressure distribution, turbulence structure, and energy losses within the jump, making air-water interaction an intrinsic component of its operating mechanism rather than a secondary effect.

Stepped spillways provide another prominent and extensively studied example of air-water interaction in hydraulic engineering. As water flows down a stepped profile, free nappes detach from step edges, impact downstream steps, and create recirculating cavities that act as efficient air-entrainment zones. At higher discharges, the flow transitions to a skimming-flow regime, in which a coherent high-speed water stream glides over a series of air-filled cavities. In this regime, air is continuously entrained and transported along the structure, and shear at the air-water interface within the cavities plays a central role in governing energy dissipation, pressure fluctuations, and flow resistance (Falvey, 1980; Chanson, 1995; Pfister and Hager, 2010). Air entrainment also provides crucial protection against cavitation by reducing pressure extremes near solid boundaries, further highlighting the dynamic role of the air phase.

Air-water interaction is likewise essential in a broad range of other hydraulic structures involving flow impingement, separation, or rapid acceleration. In plunging jets, drop shafts, and energy dissipation basins, water jets entrain large volumes of air upon impact with downstream pools, generating highly aerated bubbly flows that enhance mixing and energy loss (Falvey, 1980; Chanson, 2009). In stilling basins equipped with baffle blocks or chute blocks, flow separation and wake formation promote intense air entrainment and create highly aerated recirculation zones in which air modifies turbulence intensity, shear stresses, and pressure fields. In gated structures, such as weirs, sluice gates, and orifices, flow contraction and jet formation generate low-pressure regions that draw air from the atmosphere, influencing discharge coefficients and pressure distributions (Bos, 1989).

In open-channel flows over steep or rough beds, air entrainment occurs once turbulence levels near the free surface exceed a critical threshold. This is commonly observed in high-gradient channels, stepped channels, and macro-roughness flows, where entrained air alters flow resistance, reduces near-bed shear stress, and modifies the effective flow structure (Chanson, 1997). Similarly, in spillways and chutes, air entrainment becomes increasingly important with increasing discharge and slope, fundamentally affecting both hydraulic performance and structural safety.

What unifies these diverse hydraulic structures is that air is introduced primarily from the atmosphere through free surfaces, cavities, or openings, and once entrained or confined, it actively participates in the momentum and energy balance of the system. Depending on geometry and operating conditions, air may be dispersed within the water as bubbles, trapped in recirculating cavities, or confined above the free surface as a continuous phase. In all cases, interfacial shear, turbulence-induced exchange, and pressure gradients

generated by confinement or acceleration enable the air phase to influence hydraulic resistance, pressure fields, and flow stability.

While the role of air in hydraulic jumps, stepped spillways (Bentalha and Habi, 2019; Bentalha and Habi, 2020; Lebdiri et al., 2020), and aerated open-channel flows is well recognized and extensively documented, these studies typically focus on air entrainment rates, void fraction distributions, turbulence modulation, and energy dissipation. They seldom address configurations in which air remains as a longitudinally continuous phase dynamically coupled to the flow under steady conditions. Nevertheless, they provide clear physical evidence that even a low-density air phase can exert a measurable influence on hydraulic behavior when combined with strong turbulence or geometric confinement. These observations form an important physical backdrop for the study of air-water coupling in partially filled conduits, where air is likewise supplied from the atmosphere, confined by geometry, and induced into motion by water flow, albeit in a fundamentally different and less explored hydraulic configuration.

Because air-water interaction is regime dependent, much attention was devoted to identifying and classifying flow patterns. Large experimental datasets led to empirical flow-pattern maps that organized observations of stratified, wavy, intermittent, slug, annular, and dispersed regimes in horizontal circular pipes (Mandhane et al., 1974). These maps highlighted that the geometry of the interface, and thus air-water coupling, changes qualitatively with superficial velocities. Mechanistic transition models then emerged to explain why and when these regimes appear. A seminal contribution was the stability-based framework that predicts transitions by balancing gravity, inertia, and shear, particularly for stratified flow evolving toward wave growth and slugging (Taitel and Dukler, 1976). Subsequent unified approaches extended this framework to a wider range of inclinations and geometries, reinforcing the idea that interfacial momentum exchange controls regime evolution (Barnea, 1980). Because many practical pipelines are inclined rather than strictly horizontal, empirical-mechanistic correlations were developed to estimate holdup and pressure gradient across angles, explicitly incorporating regime effects (Beggs and Brill, 1973). To cover a broad range of fluids and operating conditions, generalized pressure-drop correlations were proposed that synthesized large datasets into compact engineering formulas (Friedel, 1979).

Beyond correlations, a more physically explicit modeling philosophy emerged, aiming to solve momentum balances for each phase with closures for wall and interfacial shear. This approach is comprehensively summarized in mechanistic modeling texts that treat gas-liquid flow as a coupled system governed by geometry, slip, and regime-dependent interfacial structure (Shoham, 2006). Within this context, stratified and stratified-wavy air-water flow in horizontal circular pipes became a focal topic because it represents the simplest configuration where air-water coupling appears through a continuous interface rather than dispersed bubbles or annuli. Detailed studies demonstrated that interfacial waves can dramatically increase interfacial shear stress, making wave-induced form drag comparable to or larger than viscous shear (Andritsos and Hanratty, 1987a). Companion studies showed that these instabilities are intrinsic to horizontal gas-liquid systems and are key drivers of regime transition (Andritsos and Hanratty, 1987b). Building on this understanding, predictive methods were developed to estimate pressure gradient and

liquid level in stratified flow by explicitly modeling the interface as a momentum-exchanging boundary (Spedding and Hand, 1997). Numerical simulations further demonstrated how wall friction and interfacial shear redistribute with changing holdup and wave state, emphasizing the sensitivity of coupled momentum balances to geometry and interface dynamics (Newton and Behnia, 2001). More recent studies continue to investigate stratified air-water flow in circular pipes using modern numerical tools and refined closures, confirming that interfacial friction behaves like an effective roughness that depends strongly on flow regime (Passoni et al., 2023).

In gravity-driven circular conduits operating under partially filled conditions (Achour and Amara, 2020a; 2020b), the presence of air above the free surface is intrinsic to the hydraulic configuration and cannot be neglected a priori. Whenever the flow depth is less than the conduit diameter, a continuous gas phase occupies the upper portion of the cross-section and forms a longitudinally connected headspace. This air originates primarily from the surrounding atmosphere and is introduced through open boundaries such as upstream free surfaces, manholes, ventilation shafts, junction structures, and overflow sections, which are ubiquitous in sewer, stormwater, and drainage networks (Falvey, 1980; Chanson, 2009). Additional contributions may arise from dissolved air released from the water under reduced pressure or intense turbulence, but atmospheric supply generally dominates under steady gravity-driven conditions (Wright et al., 2011).

Once present, the air phase cannot generally be assumed to remain static. The water flow, driven downslope by gravity, exerts shear stress along the air-water interface, inducing air motion in the longitudinal direction. This mechanism is well established in stratified gas-liquid pipe flows, where interfacial shear is recognized as a primary pathway for momentum exchange between phases (Andritsos and Hanratty, 1987a; Shoham, 2006). Even in the absence of an imposed air discharge, interfacial drag generated by the moving water is sufficient to set the air layer in motion, producing a coupled two-phase system rather than a purely open-channel configuration.

Because the air phase is confined by the conduit walls, any sustained air motion requires the establishment of a longitudinal pressure gradient to overcome resistance due to air-wall friction and interfacial shear. This pressure gradient, in turn, acts back on the water phase through the interface, modifying the longitudinal force balance of the liquid. The resulting interaction therefore constitutes a genuine momentum coupling mechanism: water motion induces air motion through interfacial shear, while air motion feeds back onto the water through pressure-gradient forcing. Similar pressure-mediated feedback mechanisms have been observed and analyzed in confined air-water systems during transient events in pipelines and tunnels, where air pockets strongly influence water motion despite the low density of air (Zhou et al., 2002; Vasconcelos and Wright, 2006). Although these studies focus primarily on unsteady phenomena, they clearly demonstrate the capacity of confined air to influence liquid momentum through pressure effects.

In partially filled circular conduits, the geometry plays a decisive role in amplifying this coupling mechanism. As the filling rate increases, the air headspace contracts rapidly, causing air velocity to rise sharply for a given induced air discharge. This geometric confinement intensifies both air-wall friction and interfacial shear, increasing the air

pressure gradient required to sustain motion. As a result, even though the air density is several orders of magnitude smaller than that of water, the pressure gradient transmitted through the interface may become non-negligible near high filling rates. This geometric amplification mechanism is analogous to the sensitivity observed in stratified gas-liquid pipe flows, where small changes in holdup or interface position can produce large variations in shear and pressure loss (Spedding and Hand, 1997; Newton and Behnia, 2001).

The air-water interface itself is central to this interaction. In steady uniform flow, the interface behaves as an active momentum-transfer boundary rather than a passive free surface. Relative motion between the two phases generates interfacial shear proportional to slip velocity, and this shear must be balanced by pressure forces in the air phase. Importantly, the interfacial shear stress and the air pressure gradient are not independent effects: the pressure gradient arises precisely because both interfacial and wall shear act on the air, and this same gradient exerts a longitudinal force on the water. This inseparability of interface shear and pressure feedback is a key feature of confined stratified flows and distinguishes them from classical open-channel flows, where the air above the free surface remains at nearly uniform atmospheric pressure and does not participate dynamically in the momentum balance (Brocchini and Peregrine, 2001).

The resulting operating mechanism differs fundamentally from that assumed in most uniform-flow hydraulic formulations. In classical open-channel theory, the air phase is implicitly passive, contributing neither shear nor pressure-gradient effects. Conversely, most gas-liquid pipe-flow studies consider pressurized systems in which both phase flow rates are imposed and the pressure gradient is an output quantity, rather than an internal coupling variable (Lockhart and Martinelli, 1949; Chisholm, 1967; Taitel and Dukler, 1976). In partially filled circular conduits, however, the air discharge is typically induced by the water flow rather than externally prescribed, and the air pressure gradient emerges naturally from the coupled momentum balances. This places such systems outside the scope of both traditional open-channel hydraulics and classical two-phase pipeline models.

The importance of air-water coupling is therefore strongly regime dependent. At low filling rates, the air region is large, air velocities are small, and the induced pressure gradients are negligible, so that the flow behaves essentially as a single-phase open-channel system. As the filling rate increases toward closure, air confinement intensifies, air velocities and pressure gradients increase, and coupling effects become potentially significant. Finally, when the conduit becomes fully pressurized, the air headspace disappears, interfacial effects vanish, and the system must reduce smoothly to single-phase pipe flow. Any physically consistent model must capture this progression and reproduce the correct limiting behavior.

These physical considerations motivate the modeling approach adopted in the present paper. Rather than treating air effects as empirical corrections or transient anomalies, the air phase is treated as an induced but dynamically active component of steady uniform flow. Separate momentum equations are formulated for water and air, coupled through interfacial shear and mediated by a confined air pressure gradient that arises naturally

from frictional resistance. By retaining the exact circular geometry and explicitly accounting for the feedback of air motion on water momentum, the proposed formulation provides a mechanistic description of the operating air-water coupling in partially filled circular conduits that is consistent with known multiphase flow physics and classical hydraulic limits, while addressing a configuration that has not been systematically treated in the existing literature.

While this extensive literature addresses air-water coupling in circular pipes, its dominant perspective is that of pressurized or controlled two-phase transport, where both phase flow rates are imposed and the pressure gradient is the main quantity of interest. A parallel but distinct literature exists in hydraulic and environmental fluid mechanics, where air-water interaction occurs primarily at free surfaces. In hydraulic structures such as spillways and chutes, air entrainment modifies effective density, turbulence, and resistance, motivating early comprehensive syntheses on air-water flow effects in open systems (Falvey, 1980). Later work emphasized scale effects and the difficulty of extrapolating laboratory observations to prototypes when air entrainment and turbulence interact strongly (Chanson, 2009). From a theoretical standpoint, advances in free-surface turbulence modeling clarified how strong turbulence can modify the boundary conditions at an air-water interface, providing a consistent framework for treating the interface as an active participant in momentum transfer (Brocchini and Peregrine, 2001). Recent reviews underline that air-water flows occur across nearly all hydraulic engineering applications and that the dominant coupling mechanisms vary widely, from entrainment and aeration in open channels to shear-driven interaction in confined flows. However, these hydraulic-structure studies typically do not involve a confined air headspace with a longitudinal pressure gradient acting back on the water.

Another important body of work concerns air in water pipelines, sewers, and tunnels, where air-water interaction can become critical during unsteady events. Classic transient-flow analyses establish the governing equations for pressurized pipes and provide the basis for understanding how compressibility and boundary conditions shape pressure waves (Wylie et al., 1993). Experiments on rapidly filling pipes containing trapped air showed that air compression and release can strongly influence water motion and pressure, even when the air volume is small (Zhou et al., 2002). In stormwater tunnels and large sewer systems, investigations identified mechanisms of air-pocket entrapment and release and linked them to hazards such as geysering, emphasizing that confined air can dominate system behavior during transients (Vasconcelos and Wright, 2006; Wright et al., 2011). These studies clearly demonstrate strong air-water coupling in circular conduits, but the coupling is primarily transient, local, and compressibility-driven rather than steady and uniform.

Taken together, the literature shows that air-water coupling has been intensively studied in (1) two-phase transport in circular pipes with imposed phase flow rates (Lockhart and Martinelli, 1949; Chisholm, 1967; Mandhane et al., 1974; Taitel and Dukler, 1976; Barnea, 1980; Beggs and Brill, 1973; Friedel, 1979; Shoham, 2006; Andritsos and Hanratty, 1987a; Andritsos and Hanratty, 1987b; Spedding and Hand, 1997; Newton and Behnia, 2001; Passoni et al., 2023), (2) air-water interaction in open-channel and hydraulic-structure flows dominated by entrainment and free-surface turbulence (Falvey,

1980; Chanson, 2009; Brocchini and Peregrine, 2001), and (3) confined air effects in pipelines and tunnels during rapid transients (Wylie et al., 1993; Zhou et al., 2002; Vasconcelos and Wright, 2006; Wright et al., 2011). Reviews of flow-pattern research further confirm that while two-phase flow in circular pipes is a mature field, most studies emphasize regime identification, pressure drop, and holdup rather than steady uniform-flow resistance in partially filled conduits (Rouhani and Sohal, 1983). What is notably absent from this otherwise rich body of work is a systematic treatment of steady, gravity-driven, partially filled circular flow in which the air phase forms a continuous headspace whose motion and friction generate a longitudinal pressure gradient that feeds back onto the water momentum balance and alters the classical discharge-filling relationship.

In addition to the regime-transition and mechanistic frameworks already discussed, further foundational studies reinforce the point that most classical work on gas-liquid interaction in circular pipes was developed for *transport-type problems* rather than for gravity-driven uniform flow in partially filled conduits. Dukler and Hubbard (1975) proposed one of the earliest comprehensive mechanistic descriptions of slug flow evolution in horizontal and near-horizontal pipes, emphasizing how interfacial shear, liquid holdup, and pressure gradient are dynamically linked through momentum balances and stability mechanisms. Although highly influential, this work, like most of the gas-liquid pipe-flow literature, treats pressure gradient as the primary outcome of interest for imposed phase flow rates rather than as a quantity that feeds back into a discharge-flow depth relationship governed by conduit geometry.

Similarly, broad syntheses of multiphase pipe-flow engineering practice stressed that gas-liquid coupling in pipelines must be modeled through separate momentum equations for each phase and carefully selected interfacial closures, but they also implicitly assume pressurized or quasi-pressurized operation where gravity plays a secondary role (Tattersson, 1991). These syntheses consolidate the understanding that interfacial shear and phase interaction dominate resistance, yet they do not address the specific hydraulic problem of steady uniform flow in partially filled circular pipes with a moving air headspace. Together with earlier reviews of flow-pattern research (Rouhani and Sohal, 1983), these works reinforce the conclusion that, while the physics of air-water coupling in circular pipes is well understood in a general sense, its formulation as a uniform-flow hydraulic resistance problem has not been systematically developed in the existing literature.

The work reported in the present paper fills a distinct gap in the existing literature, namely the absence of a steady uniform-flow framework that quantifies the influence of confined headspace air motion and interfacial shear on the discharge-filling rate relationship of partially filled circular pipes. Building on the conceptual foundation of one-dimensional two-phase momentum modeling (Wallis, 1969; Ishii and Hibiki, 2011) and on the recognition that interfacial shear and confined air pressure gradients can act together, the authors formulate coupled steady momentum equations for water and air in a partially filled circular conduit. The approach retains exact circular geometry for wetted area, perimeters, and interface length, acknowledging the strong nonlinearity inherent even in single-phase uniform flow. By explicitly solving the air momentum balance for the air pressure gradient, accounting for air-wall friction and interfacial shear, and substituting

this gradient into the water momentum equation, the authors obtain a fully coupled uniform-flow equation. The resulting formulation naturally reorganizes into a generalized hydraulic resistance kernel, in which the classical single-phase circular-conduit resistance emerges as a baseline term and additional terms represent air-water coupling through both interface shear and air-pressure feedback. Physically, the formulation clarifies why coupling can become significant near high filling rates despite the low density of air: the reduction of headspace area amplifies air velocity, friction, and pressure gradients, which then act on the water phase. Mathematically, the model is compact yet transparent, preserving action-reaction consistency at the interface and reducing to the classical uniform-flow law when air effects are inactive. In this sense, the study advances the literature by translating the general ideas of two-phase momentum coupling into a geometry-specific, steady uniform-flow framework for partially filled circular pipes, a configuration that, despite its practical relevance in hydraulic and sewer systems, has not been systematically addressed in earlier air-water flow studies.

GEOMETRIC CONSIDERATIONS

Fig. 1 shows a cross-sectional geometry of a partially filled circular pipe. The figure illustrates the angular parameterization used to describe the air-water configuration in the conduit. The water phase occupies the lower portion of the pipe, bounded by the circular wall and the air-water interface, while the air phase occupies the upper headspace. The filling state is characterized by the angle θ , measured at the pipe center, which uniquely defines the water depth y , the wetted area, the wetted perimeter, and the air-water interface length. This geometric description provides an exact and continuous representation of the conduit cross-section as a function of filling rate and constitutes the basis for deriving the hydraulic and momentum-balance relationships for coupled air-water uniform flow in circular pipes.

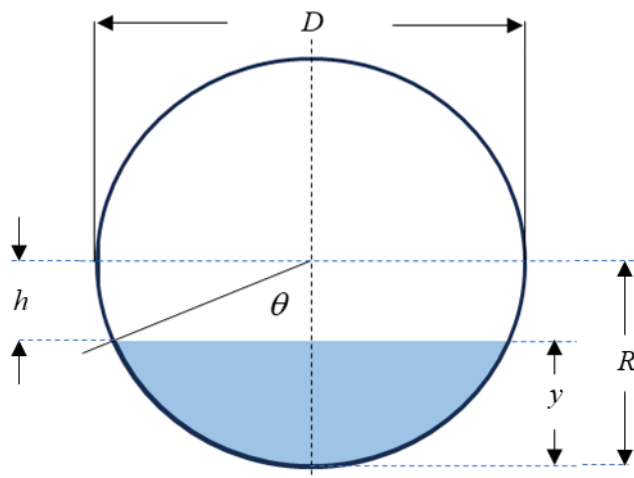


Figure 1: Cross-sectional geometry of a partially filled circular pipe

Definitions and indices

Indices (subscripts)

- Subscript w: water (liquid region)
- Subscript a: air (gas headspace region)
- Subscript i: interface (air-water free surface)

These indices indicate where a quantity acts, water, air, or interface.

Independent coordinate and constants

- Longitudinal coordinate along the conduit
 x [m]
- Gravitational acceleration
 g [m/s²]

Conduit parameters

- Diameter of the circular conduit
 D [m]
- Geometric radius
 $R = D / 2$ [m]
- Bed slope (assumed constant)
 S_0 [-]

Fluid properties

- Water density
 ρ_w [kg/m³]
- Air density
 ρ_a [kg/m³]

Flow variables

- Water discharge

$$Q \text{ [m}^3\text{/s]}$$

- Air discharge along the headspace

$$Q_a \text{ [m}^3\text{/s]}$$

- Air pressure in the headspace (mean pressure at the interface)

$$p_a(x) \text{ [Pa]}$$

Only its gradient

$$\frac{d p_a}{d x}$$

contributes to longitudinal forcing.

Circular-geometry functions

- Flow depth

$$y \text{ [m]}$$

- Filling rate (relative flow depth)

$$\xi = y / D \text{ [-]}, \quad 0 < \xi < 1$$

- Central angle, define the half-angle by circle geometry:

$$\theta(\xi) \text{ [rad]}, \quad \theta \in [0, \pi]$$

- Distance from center to free surface chord

$$h = R - y \text{ [m]}$$

Thus,

$$\frac{h}{R} = 1 - 2\xi \text{ [-]}$$

For a chord,

$$\cos \theta = h / R$$

Hence

$$\theta(\xi) = \arccos(1 - 2\xi)$$

- The dimensionless wetted-area function

wetted area as a circular segment

According to Fig. 1, the wetted water area is as follows:

$$A_w = A_{\text{sector}} - A_{\text{triangle}} \quad [\text{m}^2]$$

For the sector area corresponding to the angle 2θ , the following can be written:

$$A_{\text{sector}} = \frac{1}{2} R^2 (2\theta) = R^2 \theta$$

Regarding the triangle area, the following can be written:

$$A_{\text{triangle}} = \frac{1}{2} R^2 \sin(2\theta)$$

Use what follows:

$$\sin(2\theta) = 2 \sin \theta \cos \theta$$

Thus,

$$A_{\text{triangle}} = R^2 \sin \theta \cos \theta$$

Therefore,

$$A_w = R^2 (\theta - \sin \theta \cos \theta)$$

Which can be written in the following final form:

$$A_w(\xi) = \frac{D^2}{4} \left[\cos^{-1}(1 - 2\xi) - 2(1 - 2\xi) \sqrt{\xi(1 - \xi)} \right]$$

Now, Define the dimensionless wetted-area function as the wetted area scaled by

$$D^2 / 4$$

so that writing the following:

$$A^* = \frac{4 A_w(\xi)}{D^2} \quad [-]$$

Which can be written as follows:

$$A^* = \cos^{-1}(1 - 2\xi) - 2(1 - 2\xi)\sqrt{\xi(1 - \xi)}$$

Thus, A^* is a pure geometry function, dimensionless, equal to the wetted area A_w expressed in units of the reference area $D^2/4$. It varies from 0 (empty) to π (full).

- Air (headspace) area

$$A_a = A_{\text{conduit area}} - A_w(\xi) = \frac{\pi D^2}{4} - \frac{D^2}{4} A^*$$

Thus:

$$A_a = \frac{D^2}{4} [\pi - A^*(\xi)] \quad [\text{m}]$$

- Water-wall wetted perimeter

$$P_w = D\theta(\xi) \quad [\text{m}]$$

Thus,

$$P_w = D \cos^{-1}(1 - 2\xi)$$

- Air-wall perimeter

$$P_a(\xi) = P_{\text{conduit}} - P_w(\xi)$$

Then:

$$P_a(\xi) = \pi D - P_w(\xi)$$

Which can be rewritten as follows:

$$P_a(\xi) = \pi D - D \cos^{-1}(1 - 2\xi)$$

or as follows:

$$P_a(\xi) = D [\pi - \cos^{-1}(1 - 2\xi)] \quad [\text{m}]$$

- Interface length (free-surface chord)

$$P_i(\xi) = 2R \sin \theta$$

Which can be written in the following final form:

$$P_i(\xi) = 2D\sqrt{\xi(1-\xi)} \quad [\text{m}]$$

Velocities

- Mean water velocity

$$V_w = \frac{Q}{A_w(\xi)} \quad [\text{m/s}]$$

- Mean air velocity

$$V_a = \frac{Q_a}{A_a(\xi)} \quad [\text{m/s}]$$

Shear stresses and friction factors

- Water-wall shear stress

$$\tau_w = \frac{f_w}{8} \rho_w V_w^2 \quad [\text{N/m}^2]$$

where:

$$f_w \quad [-]$$

is the water-wall friction factor

- Air-wall shear stress

$$\tau_a = \frac{f_a}{8} \rho_a V_a^2 \quad [\text{N/m}^2]$$

where:

$$f_a \quad [-]$$

is the air-wall friction factor

- Interfacial shear stress (acting on the water)

$$\tau_i = \frac{f_i}{8} \rho_a (V_w - V_a)^2 \quad [\text{N/m}^2]$$

where:

$$f_i \quad [-]$$

is the interfacial friction factor.

Water and air momentum balances

Water-phase force balance per unit length

Driving force due to slope

$$\rho_w g A_w S_0 \quad [\text{N/m}]$$

Resistance by water-wall shear

$$\tau_w P_w \quad [\text{N/m}]$$

Resistance by interfacial shear

$$\tau_i P_i \quad [\text{N/m}]$$

Longitudinal force due to air pressure gradient

$$A_w \frac{dp_a}{dx} \quad [\text{N/m}]$$

Thus, the uniform-flow water balance can be written as follows:

$$\rho_w g A_w S_0 - \tau_w P_w - \tau_i P_i - A_w \frac{dp_a}{dx} = 0 \quad (1)$$

Air-phase force balance per unit length

Air pressure-gradient force

$$A_a \frac{dP_a}{dx} \quad [\text{N/m}]$$

- Air-wall shear resistance

$$\tau_a P_a \text{ [N/m]}$$

- Interfacial shear acts opposite on air (Newton's 3rd law), so it appears with opposite sign compared with water:

$$-\tau_i P_i \text{ [N/m]}$$

- Thus, the uniform-flow air balance can be written as follows:

$$-A_a \frac{dp_a}{dx} - \tau_a P_a + \tau_i P_i = 0 \quad (2)$$

So, solving for

$$\frac{dp_a}{dx}$$

Yields the following:

$$\frac{dp_a}{dx} = \frac{\tau_i P_i}{A_a} - \frac{\tau_a P_a}{A_a} \quad (4)$$

Substitute

$$\frac{dp_a}{dx}$$

into the water Eq. (1). This gives the following:

$$\rho_w g A_w S_0 - \tau_w P_w - \tau_i P_i - A_w \left(\frac{\tau_i P_i}{A_a} - \frac{\tau_a P_a}{A_a} \right) = 0 \quad (5)$$

Expanding Eq. (5) yields what follows:

$$\rho_w g A_w S_0 - \tau_w P_w - \tau_i P_i - A_w \frac{\tau_i P_i}{A_a} + A_w \frac{\tau_a P_a}{A_a} = 0 \quad (6)$$

Now, divide each member of Eq. (6) by $\rho_w g A_w$. This allows deriving the following:

$$S_0 - \frac{\tau_w P_w}{\rho_w g A_w} - \frac{\tau_i P_i}{\rho_w g A_w} - A_w \frac{\tau_i P_i}{A_a \rho_w g A_w} + A_w \frac{\tau_a P_a}{A_a \rho_w g A_w} = 0 \quad (7)$$

After simplifying, the following can be deduced:

$$S_0 = \frac{\tau_w P_w}{\rho_w g A_w} + \frac{\tau_i P_i}{\rho_w g A_w} + \frac{\tau_i P_i}{A_a \rho_w g} - \frac{\tau_a P_a}{A_a \rho_w g} = 0 \quad (8)$$

Eq. (8) can be reduced in the following final form:

$$S_0 = \frac{\tau_w P_w}{\rho_w g A_w} + \frac{\tau_i P_i}{\rho_w g A_w} + \frac{1}{\rho_w g} \left(\frac{\tau_i P_i}{A_a} - \frac{\tau_a P_a}{A_a} \right) \quad (9)$$

Now substitute the shear closures previously defined, and keep velocities explicit. This provides the following:

- Water-wall term

$$\frac{\tau_w P_w}{\rho_w g A_w} = \frac{1}{\rho_w g} \left(\frac{f_w}{8} \rho_w V_w^2 \right) \frac{P_w}{A_w} \quad (10)$$

Eq. (10) reduces to the following:

$$\frac{\tau_w P_w}{\rho_w g A_w} = \frac{f_w}{8 g} \frac{P_w}{A_w} V_w^2 \quad (11)$$

Water interface term

$$\frac{\tau_i P_i}{\rho_w g A_w} = \frac{1}{\rho_w g} \left[\frac{f_i}{8} \rho_a (V_w - V_a)^2 \right] \frac{P_i}{A_w} \quad (12)$$

Eq. (12) reduces to the following:

$$\frac{\tau_i P_i}{\rho_w g A_w} = \frac{f_i}{8 g} \frac{\rho_a}{\rho_w} \frac{P_i}{A_w} (V_w - V_a)^2 \quad (13)$$

- Air-wall contribution through pressure gradient

$$\frac{1}{\rho_w g} \left(-\frac{\tau_a P_a}{A_a} \right) = \frac{1}{\rho_w g} \left(-\frac{f_a}{8} \rho_a V_a^2 \frac{P_a}{A_a} \right) \quad (14)$$

Eq. (14) can be rewritten as follows:

$$\frac{1}{\rho_w g} \left(-\frac{\tau_a P_a}{A_a} \right) = -\frac{f_a}{8 g} \frac{\rho_a}{\rho_w} \frac{P_a}{A_a} V_a^2 \quad (15)$$

- Interfacial contribution through pressure gradient

$$\frac{1}{\rho_w g} \left(-\frac{\tau_i P_i}{A_a} \right) = \frac{1}{\rho_w g} \left[\frac{f_i}{8} \rho_a (V_w - V_a)^2 \right] \frac{P_i}{A_a} \quad (16)$$

Eq. (16) can be rewritten as follows:

$$\frac{1}{\rho_w g} \left(-\frac{\tau_i P_i}{A_a} \right) = \frac{f_i}{8 g} \frac{\rho_a}{\rho_w} \frac{P_i}{A_a} (V_w - V_a)^2 \quad (17)$$

Substituting Eqs. (11), (13), (14) and (17) into Eq. (9) yields what follows:

$$\begin{aligned} S_0 = & \frac{f_w}{8 g} \frac{P_w}{A_w} V_w^2 + \frac{f_i}{8 g} \frac{\rho_a}{\rho_w} \frac{P_i}{A_w} (V_w - V_a)^2 + \frac{f_i}{8 g} \frac{\rho_a}{\rho_w} \frac{P_i}{A_a} (V_w - V_a)^2 \\ & - \frac{f_a}{8 g} \frac{\rho_a}{\rho_w} \frac{P_a}{A_a} V_a^2 \end{aligned} \quad (18)$$

Rearranging Eq. (18) yields the following:

$$S_0 = \frac{f_w}{8 g} \frac{P_w}{A_w} V_w^2 + \frac{f_i}{8 g} \frac{\rho_a}{\rho_w} (V_w - V_a)^2 \left(\frac{P_i}{A_w} + \frac{P_i}{A_a} \right) - \frac{f_a}{8 g} \frac{\rho_a}{\rho_w} \frac{P_a}{A_a} V_a^2 \quad (19)$$

This is the complete coupled uniform-flow equation.

Air-activation closure and explicit dependence on Q

To represent “air effects become relevant beyond a threshold filling rate”, let’s define the following:

- Threshold filling rate:

$$\xi_0 \text{ [-]}$$

Determining the value of the above threshold filling rate requires a laborious calculation that is not useful to detail herein. However, the authors recommend considering the following approximate value:

$$\xi_0 \approx 0.70 \quad (20)$$

- Air induction parameter

$$\beta \text{ [-]}$$

This defines the fraction of water discharge that induces air flow, as indicated below.

- Define the air discharge as follows:

$$Q_a = \begin{cases} 0, & \xi < \xi_0 \\ \beta Q, & \xi \geq \xi_0 \end{cases} \quad (21)$$

For

$$\xi \geq \xi_0$$

Insert

$$\beta Q$$

into velocity definitions as follows:

$$V_w = \frac{Q}{A_w}, V_a = \frac{Q_a}{A_a} = \frac{\beta Q}{A_a} \quad (22)$$

Then, the split velocity is as follows:

$$V_w - V_a = Q \left(\frac{1}{A_w} - \frac{\beta}{A_a} \right) \quad (23)$$

Squaring it yields what follows:

$$(V_w - V_a)^2 = Q^2 \left(\frac{1}{A_w} - \frac{\beta}{A_a} \right)^2 \quad (24)$$

- On the other side, the following can be written:

$$V_a^2 = \left(\frac{\beta Q}{A_a} \right)^2 = \beta^2 \frac{Q^2}{A_a^2} \quad (25)$$

Rewrite Eq. (19) explicitly as follows

Substituting Eqs. (22) to (25) into Eq. (19) yields the following:

- Water-wall term

$$\frac{f_w}{8g} \frac{P_w}{A_w} V_w^2 = \frac{f_w}{8g} \frac{P_w}{A_w} \left(\frac{Q}{A_w} \right)^2 \quad (26)$$

Eq. (26) reduces to the following:

$$\frac{f_w}{8g} \frac{P_w}{A_w} V_w^2 = \frac{Q^2}{8g} \left(f_w \frac{P_w}{A_w^3} \right) \quad (27)$$

It is emphasis to note that the second member of Eq. (27) represents the slope S_0 of the conduit in the case of water alone, which can be derived from the Darcy-Weisbach relationship as follows:

$$S_0 = K_0(\xi) \frac{Q^2}{8g} \quad (28)$$

$K_0(\xi)$ can be defined as the single-phase water resistance coefficient. It reads as follows:

$$K_0(\xi) = f_w \frac{P_w}{A_w^3} \quad (29)$$

The coefficient $K_0(\xi)$ is a single-phase hydraulic resistance kernel derived directly from the Darcy-Weisbach formulation. It is a hydraulic resistance kernel, not a similarity coefficient. Although it is not dimensionless, its dimension L^{-5} is required for dimensional consistency when expressing the uniform-flow equation in discharge form. Physically, $K_0(\xi)$ measures the resistance imposed by conduit geometry and wall friction per unit discharge squared. Its strong dependence on the wetted area explains the non-monotonic structure of the discharge-flow depth relationship in circular conduits. More precisely, $K_0(\xi)$ quantifies how strongly geometry and wall friction oppose motion, independently of discharge magnitude. It is a purely geometric-hydraulic resistance density, controlled by both the water area and the wetted perimeter.:

- Interfacial + pressure-gradient interfacial term

This is represented by the second term of the right side of Eq. (19). It can be rewritten as follows:

$$\frac{f_i}{8g} \frac{\rho_a}{\rho_w} (V_w - V_a)^2 \left(\frac{P_i}{A_w} + \frac{P_i}{A_a} \right) = \frac{Q^2}{8g} \left[\frac{\rho_a}{\rho_w} f_i \left(\frac{1}{A_w} - \frac{\beta}{A_a} \right)^2 P_i \left(\frac{1}{A_w} + \frac{1}{A_a} \right) \right] \quad (30)$$

- Air-wall term (through pressure gradient)

This is the last term in the right side of Eq. (19). It can be written as follows:

$$- \frac{f_a}{8g} \frac{\rho_a}{\rho_w} \frac{P_a}{A_a} V_a^2 = - \frac{Q^2}{8g} \left(f_a \frac{\rho_a}{\rho_w} \beta^2 \frac{P_a}{A_a^3} \right) \quad (31)$$

Therefore, for

$$\xi \geq \xi_0$$

Eq. (19) can be written is the following reduced form, similar to Eq. (29):

$$S_0 = K(\xi) \frac{Q^2}{8g} \quad (32)$$

where

$$K(\xi) = f_w \frac{P_w}{A_w^3} + \frac{\rho_a}{\rho_w} f_i \left(\frac{1}{A_w} - \frac{\beta}{A_a} \right)^2 P_i \left(\frac{1}{A_w} + \frac{1}{A_a} \right) - f_a \frac{\rho_a}{\rho_w} \beta^2 \frac{P_a}{A_a^3} \quad (33)$$

This factor governs the entire discharge-flow depth relationship in the coupled model. **is** a generalized hydraulic resistance kernel, expressed in discharge form, that aggregates all mechanisms opposing the gravitational driving force in a partially filled circular conduit when both water and air are dynamically active. It is not a friction factor, or a correction coefficient, or an empirical fitting parameter.

Eq. (33) represents the sum of the following three terms.

The first term of Eq. (33)

$$K_0(\xi) = f_w \frac{P_w}{A_w^3}$$

represents the classical water-wall resistance; it represents pure single-phase hydraulic resistance; it Accounts for wall friction acting on the water only, and depends entirely on geometry and the water friction factor. It forms the baseline resistance of the system; it exists even when no air motion or air pressure effects are present, and it encodes the geometric nonlinearity responsible for the non-monotonic $Q^*(\xi)$ curve.

The second term of Eq. (33)

$$K_{i+p}(\xi) = \frac{\rho_a}{\rho_w} f_i \left(\frac{1}{A_w} - \frac{\beta}{A_a} \right)^2 P_i \left(\frac{1}{A_w} + \frac{1}{A_a} \right)$$

represents the interfacial shear + air pressure coupling; this is the most physically significant new contribution; this term arises from two inseparable mechanisms: (1) interfacial shear caused by relative motion between water and air at the free surface, and proportional to the slip velocity ($V_w - V_a$); (2) air pressure gradient which is generated because interfacial shear and air-wall shear require a pressure gradient to sustain air motion, and this pressure gradient acts back on the water.

These effects, (1) and (2), cannot be separated mathematically; they combine into a single resistance contribution.

In addition, the second term of Eq. (33) becomes important at high filling rates. As $\xi \rightarrow 1$, the following can be written:

$$A_a \rightarrow 0$$

Thus, terms involving $1/A_a$ grow rapidly, even though $\rho_a / \rho_w \ll 1$.

This explains how a low-density phase (air) can nonetheless influence water flow significantly near closure.

Third term of Eq. (33) is the air-wall friction via pressure gradient which can be written as follows:

$$K_a(\xi) = - f_a \frac{\rho_a}{\rho_w} \beta^2 \frac{P_a}{A_a^3}$$

This term represents energy loss in the air phase due to friction between air and the conduit wall. However, because this loss is transmitted through the air pressure gradient, it indirectly affects the water. The sign of this term is negative because the air pressure gradient required to overcome air-wall friction may either: oppose water motion, or

partially counterbalance interfacial drag. In the present formulation, this term reduces the net resistance added by the interface, depending on relative magnitudes. This reflects the fact that air motion can, in some configurations, locally redistribute pressure in a way that partially offsets interface resistance.

The role of $K(\xi)$ in the coupled air-water model is first a governing role. According to Eq. (32), the model reduces to the following:

$$Q(\xi) = \sqrt{\frac{8gS_0}{K(\xi)}} \quad (34)$$

Thus, $K(\xi)$ determines the following: (1) magnitude of discharge, (2) monotonicity or non-monotonicity, and (3) location of maxima and turning points.

In addition, any modification of $Q^*(\xi)$ must occur through $K(\xi)$.

For $\xi < \xi_0$, since the air discharge $Qa = 0$ and we typically take the coupling to be inactive, the model reduces to the single-phase law (water only), such that writing the following:

$$K(\xi) = K_0(\xi) = f_w \frac{P_w}{A_w^3} \quad (34)$$

This observation allows us to develop the following section.

Define Ψ as the ratio of the air-induced part to the single-phase part

The ratio Ψ is introduced to define the following relationship:

$$K(\xi) = K_0(\xi) [1 + \Psi(\xi)] \quad (35)$$

Herein:

$K_0(\xi)$ = unavoidable geometric resistance,

$\Psi(\xi)$ = relative contribution of air effects: it measures air effects relative to geometry, but $K(\xi)$ is the absolute resistance kernel.

It is worth noting that the well-known loop that can be observed in the filling rate curve of a circular conduit appears when the following happens:

$$\frac{dK_0}{d\xi}$$

grows faster than

$$A_w^{3/2}$$

The air-coupled terms modify the following:

$$\frac{dK}{d\xi} = \frac{dK_0}{d\xi} + \frac{d}{d\xi} [K_{i+p}(\xi) + K_a(\xi)]$$

Thus, air effects cannot create the loop from nothing, but they can shift its onset, steepen its descending branch, justify earlier truncation (splice point), and amplify sensitivity near full flow.

In short, one may state the following:

The factor $K(\xi)$ defined by Eq. (33) constitutes the complete hydraulic resistance kernel of the coupled air-water uniform-flow model. It generalizes the classical Darcy-Weisbach resistance by incorporating interfacial momentum exchange and air-pressure-mediated effects arising from confinement of the headspace. Although dimensional, its structure is dictated by energy balance and provides the correct proportionality between discharge squared and driving slope. The strong geometric sensitivity of $K(\xi)$ near full flow explains both the intrinsic non-monotonicity of circular-conduit conductivity and the amplification of this behavior when air dynamics become significant.

Equivalently, one may write what follows:

$$\Psi(\xi) = \frac{K(\xi) - K_0(\xi)}{K_0(\xi)} \quad (36)$$

It can be observed that if:

$$K(\xi) < K_0(\xi) \Leftrightarrow \Psi(\xi) < 0$$

Thus, the coupled air-water system offers less total resistance than the corresponding single-phase water flow. This is the key point. It is worth noting that the negative term in $K(\xi)$ comes from the third term of Eq. (33), which is negative. This term represents the energy dissipation occurring inside the air phase that is sustained by an air-pressure gradient which is partly decoupled from the water motion. The negative values of $\Psi(\xi)$ will be discussed in the next section.

From Eqs. (29) and (33), the following can be written:

$$K(\xi) - K_0(\xi) = \frac{\rho_a}{\rho_w} f_i \left(\frac{1}{A_w} - \frac{\beta}{A_a} \right)^2 P_i \left(\frac{1}{A_w} + \frac{1}{A_a} \right) - f_a \frac{\rho_a}{\rho_w} \beta^2 \frac{P_a}{A_a^3} \quad (37)$$

Thus, Eq. (36) becomes as follows:

$$\Psi(\xi) = \frac{\rho_a}{\rho_w} \frac{1}{f_w} \left[f_i \left(\frac{1}{A_w} - \frac{\beta}{A_a} \right)^2 P_i \left(\frac{1}{A_w} + \frac{1}{A_a} \right) - f_a \beta^2 \frac{P_a}{A_a^3} \right] \left(\frac{A_w^3}{P_w} \right) \quad (38)$$

Eq. (38) is exact and already explicit, but it still contains

$$A_w, A_a, P_w, P_a, P_i$$

In the next section, we substitute the circular geometry fully and simplify.

Substitute the circular geometry into $\Psi(\xi)$ and simplify to a fully dimensionless closed form

To do so, it is useful to recall the following relationships:

$$A_w(\xi) = \frac{D^2}{4} A^*(\xi)$$

$$A_a(\xi) = \frac{D^2}{4} [\pi - A^*(\xi)]$$

$$A^* = \cos^{-1}(1 - 2\xi) - 2(1 - 2\xi)\sqrt{\xi(1 - \xi)}$$

$$P_w = D \cos^{-1}(1 - 2\xi)$$

$$P_a(\xi) = D [\pi - \cos^{-1}(1 - 2\xi)]$$

$$P_i(\xi) = 2D\sqrt{\xi(1 - \xi)}$$

Compute the reciprocals of the areas

From the above-mentioned equations, the following can be derived:

$$\frac{1}{A_w(\xi)} = \frac{4}{D^2} \frac{1}{A^*(\xi)} \quad (39)$$

$$\frac{1}{A_a(\xi)} = \frac{4}{D^2} \frac{1}{[\pi - A^*(\xi)]} \quad (40)$$

Build the “difference” and “sum” terms explicitly

Write the following difference:

$$\frac{1}{A_w(\xi)} - \frac{\beta}{A_a(\xi)} = \frac{4}{D^2} \left[\frac{1}{A^*(\xi)} - \frac{\beta}{\pi - A^*(\xi)} \right] \quad (41)$$

Squaring it gives the following:

$$\left(\frac{1}{A_w(\xi)} - \frac{\beta}{A_a(\xi)} \right)^2 = \frac{16}{D^4} \left[\frac{1}{A^*(\xi)} - \frac{\beta}{\pi - A^*(\xi)} \right]^2 \quad (42)$$

Perform the following sum:

$$\frac{1}{A_w(\xi)} + \frac{1}{A_a(\xi)} = \frac{4}{D^2} \left[\frac{1}{A^*(\xi)} + \frac{1}{\pi - A^*(\xi)} \right] \quad (43)$$

Substitute perimeters

From the above-mentioned relationships, recall the following:

$$\begin{aligned} P_i(\xi) &= 2D\sqrt{\xi(1-\xi)}, \quad P_w = D \cos^{-1}(1-2\xi), \\ P_a(\xi) &= D \left[\pi - \cos^{-1}(1-2\xi) \right] \end{aligned} \quad (44)$$

Deduce the following

$$\left[A_w(\xi) \right]^3 = \frac{D^6}{64} \left[A^*(\xi) \right]^3 \quad (45)$$

Thus,

$$\frac{\left[A_w(\xi) \right]^3}{P_w} = \frac{D^5}{64} \frac{\left[A^*(\xi) \right]^3}{\cos^{-1}(1-2\xi)} \quad (46)$$

Substitute everything into Eq. (38)

- Start with the interfacial part inside Eq. (38), recalled as follows:

$$f_i \left(\frac{1}{A_w} - \frac{\beta}{A_a} \right)^2 P_i \left(\frac{1}{A_w} + \frac{1}{A_a} \right) \quad (47)$$

Using the above appropriate derived relationships, Eq. (47) becomes as follows:

$$f_i \left(\frac{1}{A_w} - \frac{\beta}{A_a} \right)^2 P_i \left(\frac{1}{A_w} + \frac{1}{A_a} \right) = f_i \left[\frac{16}{D^4} \left(\frac{1}{A^*(\xi)} - \frac{\beta}{\pi - A^*(\xi)} \right)^2 \right] \times \\ 2D\sqrt{\xi(1-\xi)} \left\{ \frac{4}{D^2} \left[\frac{1}{A^*(\xi)} + \frac{1}{\pi - A^*(\xi)} \right] \right\} \quad (48)$$

Eq. (48) reduces to the following:

$$f_i \left(\frac{1}{A_w} - \frac{\beta}{A_a} \right)^2 P_i \left(\frac{1}{A_w} + \frac{1}{A_a} \right) = \\ f_i \frac{128}{D^5} \sqrt{\xi(1-\xi)} \left(\frac{1}{A^*(\xi)} - \frac{\beta}{\pi - A^*(\xi)} \right)^2 \left(\frac{1}{A^*(\xi)} + \frac{1}{\pi - A^*(\xi)} \right) \quad (49)$$

- Now the air-wall part inside Eq. (38) is written as follows:

$$f_a \beta^2 \frac{P_a}{A_a^3} \quad (50)$$

where

A_a and P_a

are recalled as follows:

$$A_a(\xi) = \frac{D^2}{4} [\pi - A^*(\xi)] \\ P_a(\xi) = D [\pi - \cos^{-1}(1 - 2\xi)]$$

So, the following can be written:

$$\left[A_a(\xi) \right]^3 = \frac{D^6}{64} \left[\pi - A^*(\xi) \right]^3 \tag{51}$$

Thus, substituting the above appropriate equations into Eq. (50) yields the following:

$$f_a \beta^2 \frac{P_a}{A_a^3} = f_a \beta^2 \frac{64}{D^5} \frac{\left[\pi - \cos^{-1}(1 - 2\xi) \right]}{\left[\pi - A^*(\xi) \right]^3} \tag{52}$$

Thus, the ratio Ψ , expressed by Eq. (38), can be written in the following dimensionless form:

$$\Psi(\xi) = \frac{\rho_a}{\rho_w f_w} \frac{A^*{}^3}{\arccos(1 - 2\xi)} \left\{ 2 f_i \sqrt{\xi(1 - \xi)} \left(\frac{1}{A^*(\xi)} - \frac{\beta}{\pi - A^*(\xi)} \right)^2 \times \right. \\ \left. \left(\frac{1}{A^*(\xi)} + \frac{1}{\pi - A^*(\xi)} \right) - f_a \beta^2 \frac{\left[\pi - \cos^{-1}(1 - 2\xi) \right]}{\left[\pi - A^*(\xi) \right]^3} \right\} \tag{53}$$

This is the explicit closed-form $\Psi(\xi)$, with every geometric term specialized to the circular section.

The function $\Psi(\xi)$ represents the relative contribution of air dynamics to the hydraulic resistance, normalized by the baseline single-phase resistance $K_0(\xi)$. It condenses into a single dimensionless term the combined effects of air-water interfacial momentum exchange and air-wall friction transmitted through the longitudinal air-pressure gradient.

In addition, the correction function $\Psi(\xi)$ is derived under the assumption of a non-zero air region above the free surface and is therefore defined only for filling rates strictly less than unity. As $\xi \rightarrow 1$, the air phase disappears and all interfacial and air-pressure effects vanish; consequently, $\Psi(\xi)$ must tend to zero and the coupled formulation degenerates smoothly into the single-phase conductivity law. Apparent large or negative values obtained by substituting ξ extremely close to unity into the explicit expression of $\Psi(\xi)$ have no physical meaning and reflect the breakdown of the underlying assumptions rather than a real hydraulic effect.

As it will be demonstrated in the next section [Eq. (63)], the corrected conductivity takes the following form:

$$Q^*(\xi) \equiv Q_{\text{air}}^*(\xi) = \left[1 + \Psi(\xi) \right]^{-1/2} Q_{\text{single}}^*(\xi)$$

thereby preserving the classical geometry law while introducing a physically interpretable modification that becomes potentially significant only in the near-closure regime where the headspace area is small. For most filling rates Ψ is expected to be small, but it may

reach order unity close to full flow depending on the ventilation parameter β and the relative friction factors.

Dimensionless discharge

From Eqs. (32) and (35), the bed conduit slope can be written as follows:

$$S_0 = K_0(\xi) \left[1 + \Psi(\xi) \right] \frac{Q^2}{8g} \quad (54)$$

Substituting Eq. (34) into Eq. (54) yields the following:

$$S_0 = f_w \frac{P_w}{A_w^3} \left[1 + \Psi(\xi) \right] \frac{Q^2}{8g} \quad (55)$$

Solving for Q , Eq. (55) allows writing what follows:

$$Q(\xi) = \left[A_w(\xi) \right]^{3/2} (\xi) \sqrt{\frac{8gS_0}{f_w P_w}} \frac{1}{\sqrt{1 + \Psi(\xi)}} \quad (56)$$

Now, let's define a reference discharge scale as follows:

$$Q_{\text{ref}} = \left(\frac{D^2}{4} \right)^{3/2} \sqrt{\frac{8gS_0}{f_w D}} \quad (57)$$

Also, let's define the following dimensionless discharge:

$$Q^*(\xi) = \frac{Q(\xi)}{Q_{\text{ref}}} \quad (58)$$

On the other side, using the previous geometric relationships, the following can be written:

$$\frac{\left[A_w(\xi) \right]^{3/2}}{\sqrt{P_w(\xi)}} = \frac{\left(\frac{D^2}{4} A^* \right)^{3/2}}{\sqrt{D \cos^{-1}(1 - 2\xi)}} \quad (59)$$

Eq. (59) can be rewritten as follows:

$$\frac{[A_w(\xi)]^{3/2}}{\sqrt{P_w(\xi)}} = \left(\frac{D^2}{4}\right)^{3/2} \frac{[A^*(\xi)]^{3/2}}{\sqrt{D \cos^{-1}(1 - 2\xi)}} \quad (60)$$

Using Eqs. (56), (57) and (60), Eq. (58) yields what follows:

$$Q^*(\xi) = \frac{[A^*(\xi)]^{3/2}}{\sqrt{\cos^{-1}(1 - 2\xi)}} \frac{1}{\sqrt{1 + \Psi(\xi)}} \quad (61)$$

When $\Psi \rightarrow 0$, Eq. (61) gives exactly the following dimensionless single-phase (water only) geometry function:

$$Q_{\text{single}}^* = \frac{[A^*(\xi)]^{3/2}}{\sqrt{\cos^{-1}(1 - 2\xi)}} \quad (62)$$

Therefore, considering both Eq. (61) and (62), the dimensionless air-water model can be simply written as follows:

$$Q_{\text{air}}^*(\xi) = [1 + \Psi(\xi)]^{-1/2} Q_{\text{single}}^*(\xi) \quad (63)$$

Thus, the air effect enters only through the following multiplicative correction factor:

$$[1 + \Psi(\xi)]^{-1/2} \quad (64)$$

where $\Psi(\xi)$ is given explicitly by Eq. (53).

It is emphasis to point out that Eq. (63) must be applied under the following conditions:

$$Q_{\text{air}}^*(\xi) = \begin{cases} [1 + \Psi(\xi)]^{-1/2} Q_{\text{single}}^*(\xi), & \xi \leq \xi_{\text{max}} < 1 \\ Q_{\text{single}}^*(\xi), & \xi = 1 \end{cases} \quad (65)$$

where ξ_{max} is a cutoff, e.g. 0.95-0.98, before air vanishes. This is entirely standard in multiphase modeling.

Because $Q^*(\text{air})$ is governed by Eq. (65), a negative Ψ would mathematically imply a slight increase in conveyance relative to the single-phase model. However, this interpretation is physically meaningful only within the validity range of the coupled assumptions, in particular, for filling rates strictly less than unity with a non-negligible headspace. In the near-closure limit $\xi \rightarrow 1$, the headspace area vanishes and the adopted

air-flow closure may become inconsistent; therefore, Ψ must be regularized, or the coupling switched off, so that $\Psi \rightarrow 0$ and $Q^*(\text{air}) \rightarrow Q^*(\text{single})$.

Graphical representation of $\Psi(\xi)$ and physical interpretation

As an example, Fig. 2 illustrates the evolution of the dimensionless air-water interaction function $\Psi(\xi)$ as a function of the filling rate $\xi = y/D$ for a circular conduit, using Eq. (53), computed for the following data:

$$\beta = 0.01; f_a / f_w = 0.15; f_i / f_w = 0.15; \rho_a = 1.2 \text{ kg/m}^3; \rho_w = 1000 \text{ kg/m}^3$$

As it can be seen in Fig. 2, the values of $\Psi(\xi)$ remain very small, of order 10^{-4} to 5×10^{-5} , over the whole range $0.70 \leq \xi \leq 1$. This immediately indicates that, under the present conservative parameter set, the influence of air on the uniform-flow resistance is weak but finite. The air phase does not dominate the hydraulics; instead, it introduces a second-order correction to the classical single-phase water formulation. This magnitude is fully consistent with: the low-density ratio ρ_a/ρ_w , the small air-water discharge fraction $\beta = 0.01$, and moderate values of the friction ratios.

In addition, from Fig. 2, one may observe that from $\xi = 0.70$ to approximately $\xi \approx 0.87$, the function $\Psi(\xi)$ increases smoothly and monotonically. This could be physically interpreted as follows: (1) As the conduit fills, the air region becomes more confined; (2) The air velocity increases for a given β , enhancing interfacial shear between air and water, and air-induced pressure gradients along the conduit; (3) These mechanisms increase the effective hydraulic resistance felt by the water, which is why $\Psi(\xi)$ rises.

In this range, the air layer is still sufficiently developed to sustain motion, and the air-water coupling becomes progressively more active.

In addition, the curve in Fig. 2 exhibits a clear maximum of $\Psi(\xi)$ near $\xi \approx 0.87$. This maximum has a strong physical meaning; it represents the point where: (1) air confinement is strong enough to maximize air-water interaction, and (2) yet the air headspace is still sufficiently large to sustain an organized airflow.

At this point, competing effects of increasing air velocity which strengthens air effects, and decreasing air area which limits air motion, reach an optimal balance.

This behavior is not arbitrary; it reflects a real transition in the internal structure of the two-phase flow.

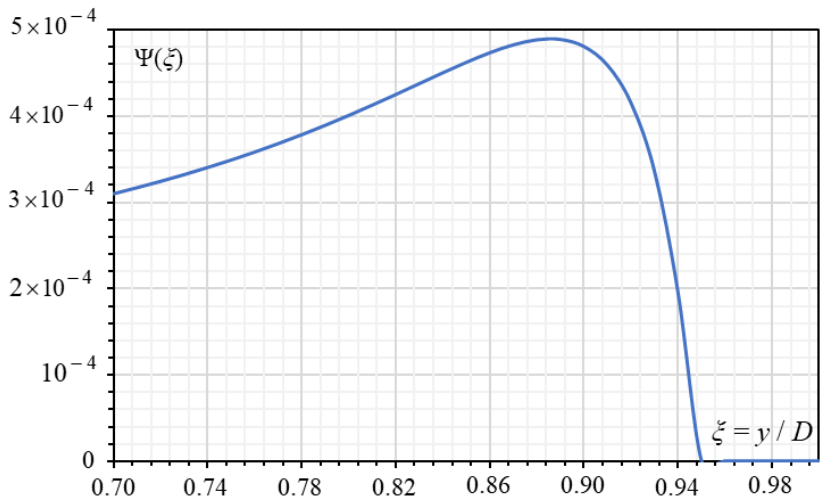


Figure 2: Variation in $\Psi(\xi)$ according to Eq. (53) for fixed values of some influencing parameters

As it can be observed, the curve in Fig. 2 exhibits a rapid decay of $\Psi(\beta)$ as $\xi \rightarrow 1$. for $\xi \geq 0.87$, the function $\Psi(\xi)$ decreases rapidly and tends to zero as $\xi \rightarrow 1$. This behavior could be physically interpreted as follows: (1) the air headspace collapses as the conduit approaches full flow; (2) The air-water interface shortens and eventually disappears; and (3) Air motion becomes increasingly restricted, and the ability of air to influence the water momentum balance vanishes.

The enforced limiting condition, recalled as follows:

$$\lim_{\xi \rightarrow 1} \Psi(\xi) = 0$$

is therefore physically mandatory and correctly reflected in Fig. 2. This ensures that, at full flow, the coupled formulation degenerates smoothly into the classical single-phase hydraulic law, with no artificial increase or decrease in conveyance.

The previous mentioned behavior has a hydraulic consequence for the discharge-flow depth relationship. Because the air-modified relative conductivity is given by Eq. (65), recalled as follows:

$$Q^*(\xi) \equiv Q_{\text{air}}^*(\xi) = [1 + \Psi(\xi)]^{-1/2} Q_{\text{single}}^*(\xi) \tag{65}$$

the strictly positive values of $\Psi(\xi)$ shown in Fig. 2 imply: (1) a slight reduction of discharge capacity relative to the single-phase model, (2) most pronounced near $\xi \approx 0.87$, and (3) vanishing as $\xi \rightarrow 1$.

This behavior is physically coherent since the air never enhances conveyance near full flow, and its influence disappears exactly when the conduit becomes fully pressurized.

The overall physical meaning of Fig. 2 can be pointed out as follows. It demonstrates that: the air effect is localized in the high-filling rates, near-closure regime, it reaches a finite maximum before diminishing, and it does not introduce any nonphysical behavior near full flow.

The resulting $\Psi(\xi)$ curve in Fig. 2 provides a mechanistically consistent explanation of how confined air can slightly modify uniform flow resistance in circular conduits, without contradicting fundamental hydraulic principles or the known single-phase limits.

In short, Fig. 2 validates the proposed coupled air-water formulation for conservative parameter values: the air effect is weak but physically interpretable, bounded, and vanishes correctly as the conduit approaches full flow.

Another example is illustrated in Fig. 3, which has been plotted according to Eq. (53) while adopting the following data.

$$\beta=0.10; f_a / f_w=0.15; f_i / f_w=0.15; \rho_a=1.2 \text{ kg/m}^3; \rho_w=1000 \text{ kg/m}^3$$

Herein, β is set to a value two times larger than in the previous case.

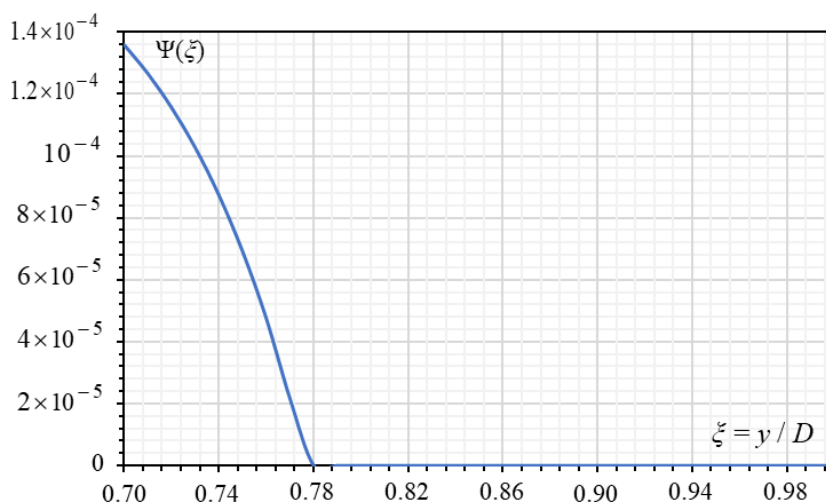


Figure 3: Variation in $\Psi(\xi)$ according to Eq. (53) for fixed values of some influencing parameters

Fig. 3 presents the same function $\Psi(\xi)$ as in Fig. 2 for a larger air induction parameter, $\beta = 0.10$, which is two times greater than in Fig. 2. This increase in β fundamentally alters the shape of the $\Psi(\xi)$ curve and, consequently, the operating mechanism of air-water coupling.

Unlike Fig. 2, Fig. 3 does not exhibit a local maximum. Instead, $\Psi(\xi)$ decreases monotonically with increasing ξ and reaches zero at a significantly lower filling rate. This behavior indicates that, for larger β , the imposed or induced air discharge becomes too

strong relative to the shrinking air headspace as the conduit fills. As a result, the air phase can no longer maintain a coherent longitudinal motion over high filling rates. The system rapidly transitions from a regime where air motion is dynamically sustained and coupled to the water, to a regime where the air headspace effectively collapses and the coupling vanishes.

Physically, doubling β increases the air velocity at any given filling rate. While this enhances interfacial shear at moderate ξ , it also accelerates the onset of geometric limitations. The rapid increase in air velocity amplifies frictional losses in the air phase and intensifies the pressure-gradient requirement, but the available air cross-section decreases too quickly to sustain these effects as ξ increases. Consequently, the regime characterized by balanced amplification and confinement, which produces a maximum in Fig. 2, no longer exists, and $\Psi(\xi)$ decays monotonically.

The comparison between Figs. 2 and 3 therefore highlights a key result of the coupled model: the existence and location of a maximum in $\Psi(\xi)$ are not universal but depend critically on the air induction parameter β . For moderate β , air-water coupling is strongest near high filling rates but remains bounded and localized. For larger β , coupling is shifted toward lower filling rates and is rapidly suppressed as the conduit fills. This demonstrates that the influence of air on uniform-flow resistance is controlled not only by geometry but also by the relative magnitude of induced air motion, and it confirms that increasing β does not necessarily strengthen air effects near closure, instead, it can eliminate them by destabilizing sustained air flow.

Overall, the two figures together provide a clear and mechanistically consistent illustration of how air induction intensity governs the structure of air-water coupling in partially filled circular conduits. They confirm that air effects remain limited, bounded, and geometry-controlled, while offering valuable insight into the parameter regimes in which air-water interaction is most likely to influence uniform-flow hydraulics.

CONCLUSION

This study has addressed a long-standing but largely overlooked problem in hydraulic modeling: the role of confined air motion and air-induced pressure gradients on steady uniform flow in partially filled circular conduits. While the hydraulics of single-phase flow in circular pipes are well established, and two-phase air-water interactions have been extensively investigated in pressurized pipelines and unsteady filling scenarios, a consistent framework describing steady, gravity-driven air-water coupling in partially filled conduits was previously lacking. The present work fills this gap by developing a mechanistic and geometry-explicit model that rigorously accounts for air-water momentum interaction under uniform-flow conditions.

The core contribution of the study lies in the formulation of coupled one-dimensional momentum balances for the water phase and the confined air headspace. By explicitly resolving the air momentum equation and eliminating the air pressure gradient through force equilibrium, the analysis yields a fully coupled uniform-flow equation that

preserves physical consistency at the air-water interface. Retaining exact circular geometry throughout the derivation allows the strong geometric nonlinearity inherent to partially filled conduits to be captured without approximation. The resulting formulation naturally reorganizes into a generalized hydraulic resistance kernel, expressed in a relative discharge form, which extends the classical Darcy-Weisbach framework to include interfacial shear and air-pressure-mediated feedback mechanisms.

A key conceptual advance of the paper is the identification and interpretation of the generalized resistance kernel $K(\xi)$ as the governing quantity controlling conveyance, monotonicity, and turning points in the discharge-filling relationship. This kernel aggregates three distinct but inseparable contributions: classical water-wall resistance, interfacial shear combined with air-pressure feedback, and air-wall friction transmitted through the air pressure gradient. Expressing the coupled problem in this form clarifies that air effects do not act as arbitrary correction factors but as physically grounded resistance mechanisms whose magnitude is dictated by geometry, friction factors, and induced air motion.

The introduction of the dimensionless interaction function $\Psi(\xi)$ further provides a compact and physically interpretable measure of the relative importance of air dynamics compared with baseline single-phase resistance. Analytical development and numerical evaluation of $\Psi(\xi)$ demonstrate that air effects are generally weak but finite, strongly geometry-controlled, and localized near high filling rates where headspace confinement becomes significant. Importantly, the model reproduces the correct limiting behavior: air-water coupling vanishes smoothly as the conduit approaches full flow, and the formulation degenerates exactly to the classical single-phase hydraulic law, ensuring consistency with established theory.

The parametric analysis reveals that the structure of air-water coupling is highly sensitive to the air-induction parameter. For moderate air induction, $\Psi(\xi)$ exhibits a well-defined maximum at high filling rates, corresponding to an optimal balance between air confinement and the ability of the headspace to sustain organized flow. For larger air induction, this balance is disrupted, the maximum disappears, and coupling effects collapse at lower filling rates. This behavior highlights a central physical insight of the study: increasing air motion does not necessarily amplify air effects near closure; instead, excessive air induction may destabilize sustained coupling by overwhelming geometric constraints.

From a hydraulic perspective, the results show that air dynamics can slightly modify conveyance capacity and the shape of the discharge-filling relationship but cannot create nonphysical behavior or dominate gravity-driven flow. Air effects may shift, amplify, or truncate existing geometric features of the circular-conduit conductivity curve, yet they remain bounded and subordinate to the fundamental geometric resistance imposed by the conduit. This finding reconciles intuitive engineering expectations with rigorous momentum analysis and explains why air effects are often subtle but potentially relevant in near-closure regimes.

The proposed framework bridges the conceptual gap between open-channel hydraulics, where air is assumed dynamically passive, and two-phase pipeline theory, where both phase discharges are imposed. By treating air motion as induced rather than prescribed, and by embedding air-water coupling directly into a steady uniform-flow formulation, the model provides a new mechanistic basis for analyzing partially filled circular conduits encountered in sewer systems, stormwater networks, and drainage infrastructure.

Future work should focus on experimental validation of the proposed resistance kernel, refinement of the air-induction closure for different ventilation conditions, and extension of the framework to unsteady flows, inclined conduits, and non-circular geometries. Nevertheless, the present study establishes a coherent theoretical foundation for steady air-water coupling in partially filled circular conduits and demonstrates that such coupling can be incorporated into hydraulic modeling in a physically transparent, mathematically consistent, and practically interpretable manner.

Declaration of competing interest

The author declares that he has no known competing financial interests or personal relationships that could influence the work reported in this article.

REFERENCES

- ACHOUR B., AMARA L., MEHTA D., BALAGANESAN P. (2022a). Compactness of hydraulic jump rectangular stilling basins using a broad-crested sill, *Larhyss Journal*, No 51, pp. 31-41.
- ACHOUR B., AMARA L. (2020a). New theoretical considerations on the critical flow in a circular conduit (Part 2), *Larhyss Journal*, No 44, pp. 31-41.
- ACHOUR B., AMARA L. (2020b). Proper relationship of manning's coefficient in a partially filled circular pipe, *Larhyss Journal*, No 42, pp. 107-119.
- ACHOUR B., AMARA L., MEHTA D. (2022b). Control of the hydraulic jump by a thin-crested sill in a rectangular channel - new experimental considerations, *Larhyss Journal*, No 50, pp. 31-48.
- ACHOUR B., AMARA L. (2023). Control of the hydraulic jump by a broad-crested sill in a rectangular channel – new theory and experiment, *Larhyss Journal*, No 54, pp. 145-174.
- AMARA L., LACHI H., BERREKSI A. (2019). Numerical computation of hydraulic jump by finite element model, *Larhyss Journal*, No 38, pp. 59-70. (In French)
- ANDRITSOS N., HANRATTY T.J. (1987a). Influence of interfacial waves in stratified gas-liquid flows, *American Institute of Chemical Engineers Journal (AIChE Journal)*, Vol. 33, Issue 3, pp. 444-454.
<https://doi.org/10.1002/aic.690330311>

- ANDRITSOS N., HANRATTY T.J. (1987b). Interfacial instabilities for horizontal gas-liquid flows in pipelines, *International Journal of Multiphase Flow*, Vol. 13, Issue 5, pp. 583-603.
[https://doi.org/10.1016/0301-9322\(87\)90014-7](https://doi.org/10.1016/0301-9322(87)90014-7)
- BARNEA D. (1980). A unified model for flow-pattern transitions for gas-liquid flow in horizontal and inclined pipes, *International Journal of Multiphase Flow*, Vol. 6, Issue 3, pp. 217-225.
[https://doi.org/10.1016/0301-9322\(80\)90051-1](https://doi.org/10.1016/0301-9322(80)90051-1)
- BEGGS H.D., BRILL J.P. (1973). A study of two-phase flow in inclined pipes, *Journal of Petroleum Technology*, Vol. 25, Issue 5, pp. 607-617.
<https://doi.org/10.2118/4007-PA>
- BENABDESSELAM A., HOUICHI L., ACHOUR B. (2017). Hydraulic jumps in a straight rectangular compound channel: theoretical approach and experimental study, *Larhyss Journal*, No 29, pp. 323-340.
- BENTALHA C., HABI M. (2019). Numerical study of the flow field for moderate slope stepped spillways, *Larhyss Journal*, No 37, pp. 39-51.
- BENTALHA C., HABI M. (2020). Inception point and air-water flow characteristics over stepped spillway: numerical study, *Larhyss Journal*, No 41, pp. 91-105.
- BROCCHINI M., PEREGRINE D.H. (2001). The dynamics of strong turbulence at free surfaces, Part 2, Free-surface boundary conditions, *Journal of Fluid Mechanics*, Vol. 449, pp. 255-290.
<https://doi.org/10.1017/S0022112001006322>
- BOS M.G. (1989). *Discharge Measurement Structures*, Book, 3rd Edition, Wageningen: International Institute for Land Reclamation and Improvement (ILRI), The Netherlands.
- BOURICHE F., DEBABECHE M., CARVALHO RITA F., DJEDDOU M. (2023). Velocity distribution in a controlled hydraulic jump in a compound channel: an experimental and machine learning (ML) study, *Larhyss Journal*, No 54, pp. 217-237.
- CHANSON H. (1995). Air entrainment in skimming flow on stepped spillways, *Canadian Journal of Civil Engineering*, Vol. 22, Issue 3, pp. 816-822.
<https://doi.org/10.1139/195-097>
- CHANSON H. (1997). *Air-Bubble Entrainment in Free-Surface Turbulent Shear Flows*, Research monograph / advanced textbook, Academic Press Publisher, London, UK, United Kingdom, 401p.
- CHANSON H. (2009). Turbulent air-water flows in hydraulic structures: dynamic similarity and scale effects, *Environmental Fluid Mechanics*, Vol.9, Issue 2, pp. 125-142.
<https://doi.org/10.1007/s10652-008-9078-3>

- CHISHOLM D. (1967). A theoretical basis for the Lockhart-Martinelli correlation for two-phase flow, *International Journal of Heat and Mass Transfer*, Vol. 10, Issue 12, pp. 1767-1778.
[https://doi.org/10.1016/0017-9310\(67\)90047-6](https://doi.org/10.1016/0017-9310(67)90047-6)
- DUKLER A.E., HUBBARD M.G. (1975). A model for gas-liquid slug flow in horizontal and near-horizontal tubes, *Industrial & Engineering Chemistry Fundamentals*, Vol. 14, Issue 4, pp. 337-347.
<https://doi.org/10.1021/i160056a011>
- FALVEY H.T. (1980). *Air-Water Flow in Hydraulic Structures*, Engineering Monograph No. 41, Denver, Colorado: United states Bureau of Reclamation, USA.
- FRIEDEL L. (1979). Improved friction pressure drop correlation for horizontal and vertical two-phase pipe flow, *European Two-Phase Flow Group Meeting*, Paper E2, Ispra, Italy.
- ISHII M., HIBIKI T. (2011). *Thermo-Fluid Dynamics of Two-Phase Flow*, Book, 2nd Edition, Springer Publishing, New York, NY, USA, 1076p.
- LEBDIRI F., SEGHIR A., BERREKSI A. (2020). Steps number effect on hydraulic parameters of flows in stepped spillways, *Larhyss Journal*, No 42, pp. 41-51.
- LOCKHART R.W., MARTINELLI R.C. (1949). Proposed correlation of data for isothermal two-phase, two-component flow in pipes, *Chemical Engineering Progress*, Vol. 45, Issue 1, pp. 39-48.
- MANDHANE J.M., GREGORY G.A., AZIZ K. (1974). A flow pattern map for gas-liquid flow in horizontal pipes, *International Journal of Multiphase Flow*, Vol. 1, Issue 4, pp. 537-553.
[https://doi.org/10.1016/0301-9322\(74\)90006-8](https://doi.org/10.1016/0301-9322(74)90006-8)
- NEWTON C.H., BEHNIA M. (2001). A numerical model of stratified wavy gas-liquid pipe flow, *Chemical Engineering Science*, Vol. 56, Issue 24, pp. 6857-6868.
[https://doi.org/10.1016/S0009-2509\(01\)00350-8](https://doi.org/10.1016/S0009-2509(01)00350-8)
- PASSONI S., PAGLIANTI A., MAGELLI F., SOLDATI A. (2023). Modelling and simulation of stratified gas-liquid flow in horizontal pipes, *Chemical Engineering Research and Design*, Vol. 193, pp. 411-426.
<https://doi.org/10.1016/j.cherd.2023.02.012>
- PFISTER M., HAGER W.H. (2010). Chute aerators: Hydraulic design, *Journal of Hydraulic Engineering*, American Society of Civil Engineers (ASCE), Vol. 136, Issue 6, pp. 352-361.
[https://doi.org/10.1061/\(ASCE\)HY.1943-7900.0000182](https://doi.org/10.1061/(ASCE)HY.1943-7900.0000182)
- ROUHANI S.Z., SOHAL M.S. (1983). Two-phase flow patterns: a review of research results, *Annals of Nuclear Energy*, Vol. 10, Issue 6, pp. 219-252.
[https://doi.org/10.1016/0306-4549\(83\)90019-6](https://doi.org/10.1016/0306-4549(83)90019-6)

- SHOHAM O. (2006). Mechanistic Modeling of Gas-Liquid Two-Phase Flow in Pipes, published by the Society of Petroleum Engineers, headquartered in Richardson, TX, Texas, USA.
- SPEDDING P.L., HAND N.P. (1997). Prediction in stratified gas-liquid co-current flow in horizontal pipelines, *International Journal of Heat and Mass Transfer*, Vol. 40, Issue 8, pp. 1923-1935.
[https://doi.org/10.1016/S0017-9310\(96\)00253-0](https://doi.org/10.1016/S0017-9310(96)00253-0)
- TAITEL Y., DUKLER A.E. (1976). A model for predicting flow regime transitions in horizontal and near-horizontal gas-liquid flow, *American Institute of Chemical Engineers Journal (AIChE Journal)*, Vol. 22, Issue 1, pp. 47-55.
<https://doi.org/10.1002/aic.690220105>
- TATTERSON G.B. (1991). *Fluid Mixing and Gas Dispersion in Agitated Tanks*, Book, McGraw-Hill, New York, NY, USA, 576p.
- VASCONCELOS J.G., WRIGHT S.J. (2006). Mechanisms for air pocket entrapment in stormwater storage tunnels, In *Proceedings of the World Environmental and Water Resources Congress 2006*, American Society and Civil Engineers (ASCE), May 21-25, Reston, Virginia, VA, USA.
[https://doi.org/10.1061/40856\(200\)206](https://doi.org/10.1061/40856(200)206)
- WALLIS G.B. (1969). *One-Dimensional Two-Phase Flow*, Book, McGraw-Hill, New York, NY, USA, 408p.
- WRIGHT S.J., LEWIS J.W., VASCONCELOS J.G. (2011). Geysering in rapidly filling storm-water tunnels, *Journal of Hydraulic Engineering*, American Society and Civil Engineers (ASCE), Vol. 137, Issue 1, pp. 112-115.
[https://doi.org/10.1061/\(ASCE\)HY.1943-7900.0000292](https://doi.org/10.1061/(ASCE)HY.1943-7900.0000292)
- WYLIE E.B., STREETER V.L., SUO L. (1993). *Fluid Transients in Systems*, Academic Textbook/Monograph, Prentice Hall Publisher, Englewood Cliffs, New Jersey, NJ, USA, 448p.
- ZHOU F., HICKS F.E., STEFFLER P.M. (2002). Transient flow in a rapidly filling horizontal pipe containing trapped air, *Journal of Hydraulic Engineering*, American Society of Civil Engineers (ASCE), Vol. 128, Issue 6, pp. 625-634.
[https://doi.org/10.1061/\(ASCE\)0733-9429\(2002\)128:6\(625\)](https://doi.org/10.1061/(ASCE)0733-9429(2002)128:6(625))