



DIRECT PRESSURIZED PIPE DIAMETER DETERMINATION FROM DARCY-WEISBACH AND COLEBROOK-WHITE RELATIONSHIPS

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ABSTRACT

The pipe-diameter design problem constitutes a strongly coupled nonlinear engineering system whose governing equations exhibit multiple nonlinear interactions. The proposed analytical reformulation provides an efficient nonlinear solution framework applicable to engineering systems governed by coupled nonlinear equations.

The determination of the diameter of pressurized circular pipes from the Darcy-Weisbach and Colebrook-White equations remains one of the most challenging problems in hydraulic engineering because the unknown diameter simultaneously influences the Reynolds number, the relative roughness, the Darcy friction factor, and the energy equation. This multiple nonlinear coupling has traditionally required nested iterative procedures involving successive evaluations of both the friction factor and the pipe diameter.

The present work develops a rigorous analytical framework for the direct determination of the pipe diameter without modifying either the Darcy-Weisbach or the Colebrook-White relationships. By introducing an appropriate change of variable, the original coupled hydraulic problem is transformed into a single nonlinear scalar equation involving only one unknown. The transformed equation is shown to be continuous and strictly monotonic over the physically admissible domain, thereby proving the existence and uniqueness of the positive solution.

An efficient computational strategy is then developed by combining the Rough Model Method (RMM) with Halley's third-order iterative scheme. The RMM provides a physically meaningful analytical initial estimate located in the immediate vicinity of the exact solution, while a single Halley correction is generally sufficient to recover the exact diameter with near machine precision. Numerical examples demonstrate that the

analytical approximation supplied by the RMM alone already yields exceptionally accurate pipe diameters, whereas the one-step Halley refinement reduces the remaining error to practically negligible levels.

The analytical formulation is further extended to the fully rough and hydraulically smooth turbulent-flow limits. Exact asymptotic expressions for the pipe diameter are derived in closed form using the principal branch of the Lambert W function. To avoid iterative evaluation of this special function, a highly accurate analytical approximation of W_0 is employed, leading to explicit diameter formulas exhibiting extremely small relative deviations from the exact Lambert-based solutions.

Finally, the study demonstrates that the general transformed Colebrook-White equation admits a rigorous series-based analytical treatment in which the exact Lambert W -based asymptotic solutions constitute the leading-order kernels, while the neglected resistance mechanism is progressively restored through convergent logarithmic expansions. This development establishes a unified analytical connection between the general turbulent-flow solution and its two asymptotic limits, thereby providing new insight into the mathematical structure of the Darcy-Weisbach-Colebrook-White diameter-design problem.

This final section represents the cornerstone of the study, providing the definitive methodology for the safe hydraulic design of pressurized circular conduits.

Keywords: Darcy-Weisbach equation; Colebrook-White equation; Pipe diameter determination; Analytical reformulation; Rough Model Method; RMM; Halley's method; Lambert W function; Asymptotic analysis; Series expansion; Pressurized pipe flow.

INTRODUCTION

The determination of pipe diameter constitutes a strongly coupled nonlinear engineering problem, where the governing equations are characterized by multiple interacting nonlinearities arising from the simultaneous dependence of the Reynolds number, relative roughness, friction factor, and energy equation on the unknown diameter. The proposed analytical reformulation removes this multiple coupling by reducing the original problem to a single nonlinear equation, thereby providing a rigorous and computationally efficient framework for solving engineering systems governed by coupled nonlinear equations.

The prediction of frictional energy losses in pressurized pipe systems has remained one of the fundamental problems of hydraulic engineering and fluid mechanics since the pioneering works of Weisbach (1845) and Darcy (1854). Virtually every hydraulic installation, including water-supply networks, irrigation systems, hydropower plants, industrial pipelines, oil and gas transmission lines, and cooling circuits, relies on an accurate evaluation of flow resistance to ensure both hydraulic performance and economic efficiency (Achour and Amara, 2022). Because the head loss directly governs the energy required to transport a prescribed discharge, its reliable estimation plays a

decisive role not only in the hydraulic design of conduit systems but also in their operational optimization and life-cycle cost analysis. Similar conclusions have been reported in studies devoted to hydraulic resistance and channel flow analysis (Frederiksen and De Vries, 1965). Consequently, the determination of hydraulic resistance has attracted continuous scientific interest for more than one and a half centuries and remains an active field of research. These principles constitute the basis of the design methodology adopted in modern water-conveyance engineering (Falvey, 1987).

For steady incompressible flow in circular pressurized conduits, the Darcy-Weisbach equation has become the universally accepted relationship for expressing frictional head losses. Owing to its sound physical basis, dimensional consistency, and applicability over a broad range of hydraulic conditions, it has progressively replaced many earlier empirical formulations and is now regarded as the reference equation for the analysis and design of pressurized pipe systems. Unlike empirical resistance laws developed for specific flow conditions, the Darcy-Weisbach equation provides a unified framework in which the energy dissipation is expressed through the dimensionless Darcy friction factor, thereby allowing the effects of flow regime, wall roughness, and conduit geometry to be incorporated within a single physically meaningful parameter. This generality explains why the Darcy-Weisbach formulation has become the cornerstone of modern hydraulic engineering and continues to be recommended in engineering practice as well as in international design standards and remains the principal formulation presented in modern hydraulic textbooks (Chow, 1959).

A major advance in the understanding of hydraulic resistance was achieved through the classical experiments of Reynolds (1883), who demonstrated the existence of distinct laminar and turbulent flow regimes and established the Reynolds number as the governing similarity parameter controlling the transition between them. Later, the systematic experiments of Nikuradse (1933) on artificially roughened pipes revealed the fundamental influence of wall roughness on turbulent flow resistance and provided the experimental foundation for distinguishing hydraulically smooth, transitional, and fully rough turbulent regimes. Additional investigations by Pigott (1933), Kemler (1933), Rouse (1943), Ackers (1958), Hager (1985), and Moody (1944) further clarified the combined influence of Reynolds number and relative roughness on the Darcy friction factor, ultimately leading to one of the most influential developments in hydraulic engineering.

This development culminated in the publication of the celebrated Colebrook-White equation (Colebrook and White, 1937; Colebrook, 1939), which successfully combines the effects of Reynolds number and relative roughness into a single implicit relationship for the Darcy friction factor. More than eight decades after its publication, the Colebrook-White equation remains the international reference for predicting turbulent friction losses in circular pressurized pipes because of its remarkable agreement with experimental observations over an exceptionally wide range of hydraulic conditions. Shortly thereafter, Moody (1944) transformed this implicit relationship into the well-known Moody diagram, which rapidly became one of the most widely used engineering tools for estimating friction factors in practical hydraulic design. Despite the enormous

progress achieved in computational hydraulics, both the Colebrook-White equation and the Moody chart continue to constitute the standard reference adopted in hydraulic textbooks, engineering manuals, and professional practice. The development of resistance laws also benefited from the important contributions of Ackers (1959), particularly regarding the interpretation of flow resistance under uniform flow conditions. Subsequent discussions on resistance equations further emphasized the practical importance of these formulations in hydraulic engineering (Hager, 1987).

The remarkable success of the Darcy-Weisbach formulation, however, is accompanied by an important mathematical limitation. Unlike the head-loss equation itself, which is explicit once the Darcy friction factor is known, the practical determination of the required pipe diameter is considerably more involved because the unknown diameter simultaneously affects every hydraulic quantity governing the flow. Indeed, the diameter appears explicitly in the Darcy-Weisbach equation, while at the same time controlling both the Reynolds number and the relative roughness, which in turn determine the friction factor through the implicit Colebrook-White relationship. Consequently, the diameter cannot be isolated directly, since it is embedded within a highly coupled nonlinear system in which the unknown variable appears simultaneously through several algebraic powers and inside a logarithmic function.

This multiple nonlinear coupling constitutes one of the principal mathematical difficulties encountered in hydraulic pipe design. In conventional engineering practice, the diameter is generally determined through successive trial-and-error calculations. An initial diameter is first assumed, after which the Reynolds number, the relative roughness, and the Darcy friction factor are evaluated. The resulting head loss is then compared with the prescribed hydraulic conditions, and the assumed diameter is repeatedly adjusted until convergence is achieved. Although straightforward in principle, this iterative procedure often requires numerous evaluations of the Colebrook-White equation before the desired level of accuracy is reached, particularly when high numerical precision is required or when the computation is embedded within optimization procedures or large hydraulic network simulations.

The computational burden associated with the implicit Colebrook-White equation has therefore motivated an extensive body of research over the past several decades. Since the pioneering work of Colebrook (1939), hundreds of explicit approximations have been proposed with the objective of replacing the original implicit equation by algebraic correlations requiring no iteration. Comprehensive reviews by Brkić (2011) and Zeghadnia et al. (2019) illustrate the remarkable diversity of these developments, ranging from relatively simple engineering correlations to highly sophisticated approximations involving nested logarithmic functions, rational approximants, asymptotic expansions, and transcendental expressions. Well-known contributions include the correlations proposed by Swamee and Jain (1976), Serghides (1984), Zigrang and Sylvester (1985), and numerous subsequent formulations developed to improve either computational efficiency or numerical accuracy.

Although these explicit correlations have substantially simplified the evaluation of the Darcy friction factor, they address only one component of the overall hydraulic design

problem. Their primary objective is to approximate the Colebrook-White equation itself, whereas the determination of the pipe diameter remains intrinsically coupled to the Darcy-Weisbach equation through the Reynolds number and the relative roughness. Consequently, even when an explicit approximation of the friction factor is employed, the diameter generally continues to be determined through an external iterative procedure because its influence on the governing hydraulic variables cannot be eliminated. In other words, replacing the implicit friction law by an explicit correlation does not remove the fundamental nonlinear coupling governing the diameter-design problem.

A different analytical direction emerged with the introduction of the Lambert W function into hydraulic engineering. Following the mathematical developments of Corless et al. (1996), several investigators demonstrated that implicit logarithmic resistance laws can be transformed into explicit analytical expressions involving the principal branch of the Lambert function. This approach was subsequently applied to the Colebrook-White equation and related logarithmic resistance laws, providing mathematically exact explicit expressions for the Darcy friction factor whenever the Lambert function is evaluated with sufficient numerical precision (Brkić, 2011; Achour and Amara, 2021a). More recently, improved asymptotic approximations of the principal branch W_0 have further enhanced the practical applicability of Lambert-based formulations by avoiding the repeated iterative evaluation of the special function itself while preserving near machine-precision accuracy.

Despite these significant advances, the overwhelming majority of previous investigations have concentrated almost exclusively on the determination of the Darcy friction factor. Comparatively little attention has been devoted to the direct analytical determination of the pipe diameter itself, although this quantity constitutes the principal design variable in virtually all hydraulic engineering applications. From a mathematical viewpoint, solving for the diameter is substantially more challenging than solving for the friction factor because the unknown simultaneously influences the hydraulic resistance law, the Reynolds number, the relative roughness, and the energy equation. Consequently, the governing problem is no longer represented by a single implicit equation but by a strongly coupled nonlinear system involving several mutually dependent variables.

This observation reveals an important gap in the existing literature. Although remarkable progress has been achieved in developing explicit approximations for the Colebrook-White equation, and although Lambert W -based formulations have provided elegant explicit solutions for the Darcy friction factor, a rigorous analytical framework allowing the direct determination of the pipe diameter from the exact Darcy-Weisbach and Colebrook-White equations has remained largely unexplored. In particular, no formulation has yet succeeded in eliminating the multiple nonlinear coupling while simultaneously preserving the complete mathematical and physical content of the governing equations.

The present study is developed precisely to address this gap. Its central objective is not to propose another empirical correlation for the Darcy friction factor, nor to replace the Colebrook-White equation by an approximate resistance law. Instead, the aim is to reformulate the complete Darcy-Weisbach-Colebrook-White diameter-design problem in such a way that the unknown pipe diameter becomes the only remaining variable. The originality of the approach lies in the fact that the governing equations are not simplified, linearized, or empirically modified. Rather, they are transformed analytically so as to reveal a hidden mathematical structure that is not apparent in the classical formulation.

The key step of the proposed methodology consists in introducing the reciprocal square root of the diameter as the transformed unknown. This change of variable converts the original coupled hydraulic problem into a single scalar nonlinear equation involving a fifth-degree polynomial term and one logarithmic function. All effects of discharge, friction slope, absolute roughness, kinematic viscosity, and gravity are incorporated into three physically meaningful parameters. The resulting equation remains rigorously equivalent to the initial Darcy-Weisbach and Colebrook-White relationships, but its mathematical structure becomes considerably more transparent and suitable for analysis.

This transformation has several important consequences. First, it eliminates the need for simultaneous iterations on the pipe diameter, Reynolds number, and Darcy friction factor. Second, it reduces the design problem to the search for a single positive root. Third, it allows the mathematical properties of the governing function to be examined directly. In particular, the transformed function is shown to be continuous and strictly increasing over the physically admissible domain. As a result, the diameter-design problem admits one and only one positive solution. This proof of uniqueness is essential because it establishes that the proposed formulation is not merely computationally convenient, but also mathematically well posed.

After establishing the existence and uniqueness of the physically admissible solution, the study develops an efficient computational strategy. The initial estimate is obtained from the Rough Model Method (RMM) (Achour, 2007; Achour and Sehtal, 2014, Zegait and Achour, 2024), which provides a physically consistent approximation of the required diameter over the turbulent-flow domain. Because this initial value is already located very close to the exact solution, the role of Halley's third-order method is reduced to a final correction rather than a conventional iterative search. In practice, a single Halley correction is sufficient to recover the exact diameter with machine-level accuracy. This represents a substantial improvement over traditional procedures, which generally require repeated coupled evaluations of the Colebrook-White equation and successive updates of the diameter.

A further contribution of the work concerns the derivation of exact asymptotic solutions for the two limiting turbulent regimes. In the fully rough regime, the viscous contribution becomes negligible and the governing equation reduces to a form that can be solved explicitly through the principal branch of the Lambert W function. The resulting diameter depends only on discharge, friction slope, absolute roughness, and gravity, which is fully consistent with the physical nature of fully rough turbulent flow.

Conversely, in the hydraulically smooth regime, the roughness contribution disappears from the final expression, and the diameter becomes governed by discharge, friction slope, kinematic viscosity, and gravity. This disappearance of the roughness parameter provides a direct mathematical confirmation of the hydraulically smooth limit.

The Lambert W -based asymptotic solutions also provide a bridge between exact analytical theory and practical engineering calculation. Although the fully rough and hydraulically smooth diameters can be expressed exactly in terms of W_0 , the numerical evaluation of the Lambert function may itself require an iterative procedure. To avoid this limitation, the study employs a high-accuracy analytical approximation of the principal branch W_0 . This approximation leads to explicit approximate formulas for both fully rough and hydraulically smooth pipe diameters, without iterative evaluation of the Lambert function. Numerical applications show that the resulting diameters reproduce the exact Lambert-based solutions with extremely small relative deviations, below $10^{-40}\%$, which is far beyond the accuracy required in ordinary hydraulic design.

The overall contribution of the present work is therefore fourfold. First, it provides an exact analytical reformulation of the Darcy-Weisbach-Colebrook-White diameter problem into a single scalar equation. Second, it proves the existence and uniqueness of the physically meaningful positive solution. Third, it combines the Rough Model Method with a single Halley correction to obtain the exact diameter with outstanding numerical efficiency. Fourth, it derives exact Lambert W -based asymptotic solutions and introduces an explicit approximation of W_0 for direct estimation of fully rough and hydraulically smooth pipe diameters. Taken together, these developments provide a unified analytical and computational framework for direct pressurized pipe diameter determination, combining mathematical rigor, physical consistency, and practical simplicity.

The remainder of this paper is organized as follows. The fundamental Darcy-Weisbach and Colebrook-White relationships are first recalled, and the governing equations are combined to express the diameter-design problem in terms of a single transformed variable. The mathematical properties of the resulting scalar equation are then examined, with particular emphasis on the existence and uniqueness of the physically admissible positive root. The computational procedure based on the Rough Model Method (RMM) and a single Halley correction is subsequently presented and validated through numerical examples.

Finally, exact asymptotic solutions are derived for the fully rough and hydraulically smooth turbulent regimes using the principal branch of the Lambert W function, and the accuracy of the proposed analytical approximation of W_0 is assessed for direct diameter estimation. This final section represents the cornerstone of the study, providing the definitive explicit methodology for the safe hydraulic design of pressurized circular conduits.

BASIC EQUATIONS

The analytical development presented in the following sections is founded on the fundamental equations governing steady turbulent flow in circular pressurized pipes. For the sake of completeness and to establish a consistent notation, the principal hydraulic relationships employed throughout the derivation are briefly recalled in this section. These equations constitute the mathematical basis of the proposed formulation and are subsequently combined to eliminate the friction factor and transform the original coupled problem into a single nonlinear equation involving only the unknown pipe diameter.

The formulation relies on four fundamental relationships: the Reynolds number, the mean flow velocity, the Darcy-Weisbach equation expressed in terms of the friction slope, and the Colebrook-White resistance law. Together, these equations fully describe the interaction between the hydraulic conditions, the flow resistance, and the pipe geometry. Their systematic combination forms the cornerstone of the analytical procedure developed in this study.

Unlike conventional solution techniques, which require repeated iterative evaluations of both the friction factor and the pipe diameter, the present approach exploits these governing equations to derive an equivalent mathematical formulation that is considerably more suitable for analytical manipulation. As will be demonstrated, this reformulation leads to a well-conditioned nonlinear equation possessing a unique physically admissible solution, thereby providing the basis for an efficient and robust diameter-design procedure.

For pressurized circular pipes, the Darcy-Weisbach equation is written as follows (1854; 1845):

$$\Delta h = f \frac{L}{D} \frac{V^2}{2g} \quad (1)$$

where Δh denotes the frictional head loss, L is the pipe length, D is the pipe diameter, V is the mean flow velocity, g is the acceleration due to gravity, and f is the Darcy friction factor.

Defining S_f as the friction slope, Eq. (1) becomes as follows:

$$S_f = \frac{f}{D} \frac{V^2}{2g} \quad (1a)$$

In the present case, the mean flow velocity can be expressed as follows:

$$V = \frac{4Q}{\pi D^2} \quad (2)$$

where Q is the discharge.

Substituting Eq. (2) into Eq. (1a) and rearranging yields the following:

$$S_f = \frac{8fQ^2}{g\pi^2 D^5} \quad (3)$$

Solving Eq. (3) for f results in what follows:

$$f = \frac{g\pi^2 S_f D^5}{8Q^2} \quad (3a)$$

Furthermore, in this work, the Reynolds number R (1883) and the relative roughness r are defined as follows:

$$R = \frac{VD}{\nu} \quad (4)$$

$$r = \frac{\varepsilon}{D} \quad (5)$$

Substituting Eq. (2) into Eq. (4) yields the following result:

$$R = \frac{4Q}{\pi D \nu} \quad (6)$$

Let us recall the well-known Colebrook-White equation as follows:

$$\frac{1}{\sqrt{f}} = -2\log_{10}\left(\frac{\varepsilon}{3.7D} + \frac{2.51}{R\sqrt{f}}\right) \quad (7)$$

Using Eq. (5), Eq. (7) can be rewritten in the following form:

$$\frac{1}{\sqrt{f}} = -2\log_{10}\left(\frac{r}{3.7} + \frac{2.51}{R\sqrt{f}}\right) \quad (7a)$$

The term in parentheses is the sum of two contributions.

The first contribution:

$$\frac{r}{3.7} \quad (8)$$

derives exclusively from relative roughness.

The second contribution:

$$\frac{2.51}{R\sqrt{f}} \quad (9)$$

comes from turbulence.

In other words, the Colebrook-White formula (7a) is already the sum of two mechanisms.

ANALYTICAL REFORMULATION OF THE COLEBROOK-WHITE EQUATION

The Colebrook-White equation has long been recognized as the reference relationship for predicting the Darcy-Weisbach friction factor in turbulent flow through circular pressurized pipes. Owing to its remarkable agreement with experimental observations over a wide range of Reynolds numbers and relative roughness values, it has become one of the cornerstones of modern hydraulic engineering. However, despite its physical reliability, its implicit logarithmic structure constitutes a major obstacle to direct analytical treatment.

The mathematical difficulty arises from the fact that the unknown pipe diameter simultaneously influences the Reynolds number, the friction factor, and the logarithmic resistance term. Consequently, the governing equations form a strongly coupled nonlinear system in which the unknown variable appears in several different algebraic forms. This intricate coupling has traditionally required iterative numerical procedures involving successive evaluations of both the friction factor and the pipe diameter until convergence is achieved.

The primary objective of the present reformulation is to eliminate this multiple coupling by transforming the governing equations into an equivalent mathematical problem involving a single unknown. Rather than introducing empirical approximations or simplifying assumptions, the proposed approach preserves the complete physical content of the original Darcy-Weisbach and Colebrook-White equations while seeking a mathematical representation that is more suitable for analytical investigation.

To achieve this objective, an appropriate change of variable is introduced, allowing all occurrences of the pipe diameter to be expressed through a single transformed variable. This transformation considerably simplifies the mathematical structure of the governing equation and converts the original coupled hydraulic problem into a scalar nonlinear equation. As will be demonstrated, the resulting equation possesses highly desirable mathematical properties, including continuity, monotonicity, and the existence of a unique physically admissible solution.

This reformulation constitutes the fundamental step of the proposed methodology. It establishes a rigorous analytical framework in which the diameter-design problem can be addressed without resorting to nested iterations on the friction factor and the Reynolds number. Moreover, the transformed equation provides a convenient basis for

the development of efficient numerical algorithms, as well as for the derivation of analytical approximations in limiting hydraulic regimes. Consequently, the proposed reformulation not only simplifies the mathematical treatment of the problem but also provides new insight into the intrinsic structure of the Colebrook-White equation and its role in hydraulic pipe design.

From Eq. (3a), the following relationship can be derived:

$$\sqrt{f} = \frac{\pi \sqrt{g S_f} D^{5/2}}{2\sqrt{2} Q} \quad (10)$$

Using both Eqs. (6) and (10), and after simplification, the following product is obtained:

$$R\sqrt{f} = \frac{\sqrt{2g S_f}}{\nu} D^{3/2} \quad (11)$$

Also, from Eq. (3a), or Eq. (10), the following can be written:

$$\frac{1}{\sqrt{f}} = \frac{2\sqrt{2} Q}{\pi \sqrt{g S_f} D^{5/2}} \quad (12)$$

Substituting Eqs. (11) and (12) into Eq. (7), and cancelling 2, yields the following:

$$\frac{\sqrt{2} Q}{\pi \sqrt{g S_f} D^{5/2}} = -\log_{10} \left(\frac{\varepsilon}{3.7D} + \frac{2.51\nu}{\sqrt{2g S_f} D^{3/2}} \right) \quad (13)$$

Eq. (13) can be rearranged as follows:

$$\frac{\sqrt{2} Q \ln(10)}{\pi \sqrt{g S_f} D^{5/2}} + \ln \left(\frac{\varepsilon}{3.7D} + \frac{2.51\nu}{\sqrt{2g S_f} D^{3/2}} \right) = 0 \quad (14)$$

Now, let's introduce the following variable:

$$z = \frac{1}{\sqrt{D}} \quad (15)$$

Let's also define the following parameters:

$$\alpha = \frac{\sqrt{2} Q \ln(10)}{\pi \sqrt{g S_f}} \quad (16)$$

$$\beta = \frac{\varepsilon}{3.7} \quad (17)$$

and

$$\gamma = \frac{2.51\nu}{\sqrt{2gS_f}} \quad (18)$$

Using Eqs. (15) - (18), Eq. (14) reduces to the following form:

$$\alpha z^5 + \ln(\beta z^2 + \gamma z^3) = 0 \quad (19)$$

Thus, the original diameter problem is reduced to the following scalar equation:

$$F(z) = 0 \quad (20)$$

where

$$F(z) = \alpha z^5 + \ln(\beta z^2 + \gamma z^3) \quad (21)$$

On the other hand, Eq. (21) can also be written as follows:

$$F(z) = \alpha z^5 + 2\ln(z) + \ln(\beta + \gamma z) \quad (21a)$$

The transformation introduced through Eq. (21) constitutes the key step of the present formulation. At first sight, the original problem appears highly intricate because the unknown pipe diameter simultaneously influences the Darcy-Weisbach equation, the Reynolds number, and the logarithmic term of the Colebrook-White equation through different algebraic powers. Such a multiple nonlinear coupling is generally regarded as the principal reason why the problem has traditionally been solved by nested iterative procedures.

Remarkably, the simple change of variable defined by Eq. (15) completely alters the mathematical structure of the governing equation. By introducing the reciprocal square root of the diameter as the new unknown, all occurrences of the pipe diameter are naturally transformed into integer powers of a single variable. Consequently, the governing equation assumes the remarkably compact form expressed by Eq. (21), in which the unknown appears only through a fifth-degree polynomial term and a single logarithmic function.

It is worth emphasizing that this simplification is achieved without introducing any approximation whatsoever. The transformed equation remains rigorously equivalent to the original coupled Darcy-Weisbach-Colebrook formulation and therefore preserves its complete physical content (Achour and Amara, 2020a). The apparent mathematical complexity of the original problem is not removed by simplification of the governing

equations, but rather by selecting a transformation that reveals their underlying analytical structure.

From a computational viewpoint, the resulting formulation offers several decisive advantages. First, the coupled problem involving the simultaneous determination of the pipe diameter, Reynolds number, and friction factor is reduced to the solution of a single scalar nonlinear equation. Second, the transformed equation possesses explicit first- and second-order derivatives, making it particularly well suited to high-order iterative schemes such as Halley's method. Finally, as demonstrated subsequently, the transformed equation exhibits favorable mathematical properties, including continuity, monotonicity, and the existence of a unique physically admissible solution, thereby ensuring both robustness and rapid numerical convergence.

The simplicity of Eq. (21) should therefore not be regarded merely as an algebraic convenience, but rather as the direct consequence of an appropriate mathematical representation of the physical problem. In this respect, Eq. (15) may be viewed as the fundamental transformation that unlocks the analytical structure of the coupled hydraulic equations and provides the foundation upon which the entire solution methodology developed in the present work is constructed.

PHYSICALLY ADMISSIBLE POSITIVE ROOT

Before developing an iterative solution procedure, it is essential to establish the fundamental mathematical properties of the transformed governing equation. In particular, the existence and uniqueness of a physically admissible solution must be demonstrated, since these properties guarantee that the diameter-design problem is mathematically well posed and that the computed solution possesses a clear physical meaning.

The transformed equation derived in the previous section constitutes a nonlinear scalar equation in the single variable z . Unlike the original coupled Darcy-Weisbach-Colebrook formulation, which involves the simultaneous interaction of the pipe diameter, the Reynolds number, and the friction factor, the proposed transformation concentrates the entire hydraulic problem into a single nonlinear function. Consequently, the analysis of its mathematical behavior becomes both feasible and highly informative.

From a physical viewpoint, only positive values of z are meaningful, since the transformation (15) implies that $D > 0$. Accordingly, the analysis is naturally restricted to the positive real domain. Within this interval, it will be shown that the transformed function is continuous, continuously differentiable, and strictly monotonic. These properties immediately imply that the equation admits one and only one positive root, thereby excluding the possibility of multiple physically admissible solutions.

The proof of uniqueness is of particular importance because it provides the theoretical foundation for the proposed computational procedure. Once uniqueness has been established, any convergent iterative algorithm initialized within the physically relevant

domain is necessarily attracted toward the same solution. Consequently, the convergence of the numerical process is not merely an empirical observation but follows directly from the mathematical structure of the transformed equation itself.

The analysis presented in this section therefore constitutes a key component of the proposed methodology. It not only demonstrates the well-posedness of the diameter-design problem but also explains why the subsequent application of a high-order iterative scheme, such as Halley's method, is both mathematically justified and computationally robust. The existence of a unique physically admissible positive root ultimately ensures that the proposed formulation provides a reliable and unambiguous determination of the required pipe diameter.

From Eq. (21a), the first derivative can be written as follows:

$$F'(z) = 5\alpha z^4 + \frac{2}{z} + \frac{\gamma}{\beta + \gamma z} \quad (22)$$

For $z > 0$, all terms in Eq. (22) are positive except none; therefore, the following can be written:

$$F'(z) > 0 \quad (23)$$

Thus, $F(z)$ is strictly increasing on $z > 0$, which ensures that the physically admissible positive root is unique.

ITERATIVE METHOD USED

Although the analytical reformulation developed in the preceding sections considerably simplifies the mathematical structure of the original problem, the transformed governing equation remains transcendental because it combines a polynomial term with a logarithmic function. Consequently, no elementary closed-form solution exists for the complete equation. The determination of the physically admissible pipe diameter therefore necessarily requires the application of an iterative numerical procedure.

It should be emphasized, however, that the role of the iterative algorithm in the present formulation differs fundamentally from that encountered in conventional approaches. In the classical solution of the Darcy-Weisbach and Colebrook-White equations, the iterative process is performed directly on a strongly coupled nonlinear system involving the simultaneous evaluation of the friction factor, the Reynolds number, and the pipe diameter. Such procedures generally require nested iterations and repeated updates of several interdependent variables before convergence is achieved.

In contrast, the proposed analytical reformulation completely removes this multiple coupling. The original hydraulic problem is reduced to a single nonlinear scalar equation involving only one unknown variable. Consequently, the iterative process is no longer employed to resolve the hydraulic coupling itself, but simply to determine the unique root of a well-conditioned scalar function. This constitutes a substantial

simplification of the computational problem and significantly enhances both numerical robustness and computational efficiency.

Among the various iterative techniques available for solving nonlinear equations, Halley's method has been selected in the present study. This choice is motivated by the mathematical characteristics of the transformed equation. Indeed, the governing function possesses explicit analytical expressions for both its first and second derivatives, making the implementation of Halley's algorithm straightforward and computationally inexpensive.

A major advantage of Halley's method lies in its third-order convergence. Whereas the classical Newton-Raphson method exhibits quadratic convergence, Halley's method achieves cubic convergence in the vicinity of the solution. Consequently, the number of correct significant digits increases much more rapidly after each iteration, leading to higher computational efficiency without increasing the complexity of the algorithm.

Furthermore, unlike derivative-free methods such as the secant or regula falsi algorithms, Halley's method fully exploits the analytical information available for the transformed equation. Since both $F'(z)$ and $F''(z)$ are available in simple closed forms, the additional computational effort associated with the second derivative is negligible compared with the gain in convergence rate. In practical applications, only a few iterations are generally sufficient to achieve machine-precision accuracy.

Another important feature of the present formulation is that the transformed function has already been shown to be continuous, strictly monotonic, and to possess a unique physically admissible positive root. These mathematical properties considerably strengthen the reliability of Halley's method. The iterative process is therefore performed on a function whose analytical behavior is completely understood, thereby eliminating the risk of convergence toward spurious roots or physically meaningless solutions.

Finally, the efficiency of the proposed algorithm is further enhanced by the analytical construction of the initial estimate. Instead of selecting an arbitrary starting value, the initial approximation is derived from the Rough Model Method (RMM). This physically meaningful initialization places the first iterate in the immediate vicinity of the exact solution, thereby allowing Halley's cubic convergence to be fully exploited from the very first iterations.

The combination of an analytically reformulated governing equation, a mathematically justified initial estimate, and a third-order iterative algorithm therefore constitutes a coherent and highly efficient computational strategy. Rather than relying on repeated hydraulic iterations, the proposed methodology transforms the original coupled design problem into the solution of a single, well-conditioned nonlinear equation, whose unique positive root is determined rapidly, accurately, and unambiguously.

Halley's third-order iteration is then applied to the governing Eq. (21a); it expresses as follows:

$$z_{n+1} = z_n - \frac{2F(z_n)F'(z_n)}{2[F'(z_n)]^2 - F(z_n)F''(z_n)} \quad (24)$$

From Eq. (22), the second derivative is given as follows:

$$F''(z) = 20\alpha z^3 - \frac{2}{z^2} - \frac{\gamma^2}{(\beta + \gamma z)^2} \quad (25)$$

Halley's method is especially appropriate here because $F(z)$, $F'(z)$, and $F''(z)$ are all explicit and inexpensive to evaluate. It provides cubic convergence near the solution, whereas Newton's method provides only quadratic convergence.

EFFICIENT INITIAL GUESS

The authors recommend using the initial guess derived from the Rough Model Method (RMM). The approximate diameter D_0 to be considered is given by the following RMM fundamental relationship:

$$D_0 = \psi D^* \quad (26)$$

where the dimensionless parameter ψ is called "the dimensionless correction factor of linear dimensions"; it is expressed as follows:

$$\psi = 1.35 \left[-\log \left(\frac{\varepsilon / D^*}{4.75} + \frac{8.5}{R^*} \right) \right]^{-2/5} \quad (27)$$

The Reynolds number associated with turbulent flow in a rough circular pipe of diameter D^* is denoted by R^* and is defined as follows:

$$R^* = \frac{4Q}{\pi D^* \nu} \quad (28)$$

The related diameter D^* is explicitly given as follows:

$$D^* = \left(\frac{Q^2}{2\pi^2 g S_f} \right)^{1/5} \quad (29)$$

Therefore, the Reynolds number R^* is as follows:

$$R^* = \frac{4Q}{\pi D^* \nu} \quad (6a)$$

Thus, the initial guess z_0 to be used in the iterative process expressed by Eq. (24) is as follows:

$$z_0 = \frac{1}{\sqrt{D_0}} \quad (15a)$$

It is worth emphasizing that the fundamental relationship underlying the Rough Model Method (RMM), expressed by Eq. (26), remains valid over the entire turbulent-flow domain delineated by the Moody diagram.

NUMERICAL EXAMPLES

To illustrate the practical implementation of the proposed methodology and to demonstrate its computational efficiency, a complete numerical example is presented in this section. The objective is to determine the diameter of a circular pressurized pipe from the prescribed hydraulic characteristics, namely the discharge, the friction slope, the absolute roughness, and the kinematic viscosity of the conveyed fluid.

The example follows exactly the analytical procedure developed in the preceding sections. Each computational step is performed sequentially, beginning with the evaluation of the parameters defining the transformed equation, followed by the construction of the analytical initial estimate and the subsequent application of Halley's iterative method until convergence is achieved. Finally, the corresponding pipe diameter, Reynolds number, and Darcy-Weisbach friction factor are evaluated.

The primary purpose of this example is not merely to verify the correctness of the proposed formulation, but also to demonstrate its simplicity and practical applicability. Owing to the analytical reformulation introduced in this work, the original coupled hydraulic problem is reduced to the solution of a single nonlinear scalar equation. Consequently, the entire computational procedure becomes systematic, transparent, and straightforward to implement.

Particular attention is devoted to the convergence process in order to illustrate the effectiveness of the proposed initialization strategy and the rapid convergence of Halley's third-order iterative scheme. As will be shown, only a very limited number of iterations is required to obtain machine-precision accuracy, thereby confirming both the robustness and the computational efficiency of the proposed methodology.

This numerical example therefore provides a practical validation of the theoretical developments presented in this paper and demonstrates that the proposed formulation constitutes an efficient and reliable tool for the hydraulic design of circular pressurized pipes governed by the complete Darcy-Weisbach and Colebrook-White equations.

1. Let's consider the following data:

$$Q = 0.30 \text{ m}^3/\text{s}, S_f = 0.0005, \varepsilon = 0.001 \text{ m}, \nu = 10^{-6} \text{ m}^2/\text{s}, g = 9.81 \text{ m/s}^2.$$

1.1 Solution using Halley's iterative method [Eq. (24)]

Using Eqs. (16), (17), and (18) the following constants are obtained:

$$\alpha \approx 1.404049964, \beta \approx 0.0002702703, \text{ and } \gamma \approx 0.0000080138 \quad (30)$$

Substituting these constants into the governing Eq. (21a), the following is obtained:

$$F(z) = 1.404049964 z^5 + 2 \ln(z) + \ln(0.0002702703 + 0.0000080138 z)$$

An iterative process using Halley's method [Eq. (24)] gives the following exact positive physically admissible root as follows:

$$z = 1.3982776435 \text{ m}^{-1/2} \quad (31)$$

The exact positive root of Eq. (31) was computed using Halley's third-order iterative method, Eq. (24), whose cubic convergence makes it particularly suitable for the present problem owing to the availability of explicit analytical expressions for both the first and second derivatives of the transformed function. The following convergence criterion was used:

$$|z_{n+1} - z_n| < 10^{-12} \quad (32)$$

According to Eq. (15), the sought diameter D is given as follows:

$$D = \frac{1}{z^2} \approx 0.511461 \text{ m} \quad (33)$$

1.2 Approximate solution using the initial guess [Eq. (15a)]

The exact solution presented in the preceding section demonstrates the effectiveness of the proposed analytical reformulation when combined with Halley's iterative method. Nevertheless, from a practical engineering perspective, it is of considerable interest to examine the accuracy that can be achieved by using only the analytical initial estimate provided by Eq. (15a), without performing any iterative correction.

The purpose of the present section is therefore to assess the predictive capability of the proposed initial approximation. As will be shown, the analytical estimate derived from the Rough Model Method (RMM) already provides a remarkably accurate approximation of the transformed variable, leading to pipe diameters that differ only marginally from the exact solution. This observation highlights the intrinsic quality of the proposed initial guess and demonstrates that it captures the dominant nonlinear behavior of the governing equation.

The practical implications of this result are particularly noteworthy. In many engineering applications, the approximation supplied by Eq. (15a) is sufficiently accurate to be used directly, thereby eliminating the need for iterative calculations altogether. When higher numerical precision is required, a single Halley correction is generally sufficient to recover the exact solution to machine accuracy. Consequently, the proposed methodology offers a unique combination of analytical simplicity, computational efficiency, and numerical precision, making it especially attractive for hydraulic design calculations and engineering software implementations.

Using the previous data, Eq. (29) gives the diameter D^* of the rough pipe as follows:

$$D^* = \left(\frac{Q^2}{2\pi^2 g S_f} \right)^{1/5} = 0.62180578 \text{ m} \quad (29)$$

Consequently, Eq. (6a) allows computing the Reynolds number R^* in the rough model as follows:

$$R^* = \frac{4Q}{\pi D^* \nu} = 614,294.493 \quad (6a)$$

Thus, Eq. (27) allow us to compute the dimensionless parameter ψ as follows:

$$\psi = 1.35 \left[-\log \left(\frac{\varepsilon / D^*}{4.75} + \frac{8.5}{R^*} \right) \right]^{-2/5} \approx 0.82235158 \quad (27)$$

Then, the approximate sought diameter D_0 is given by the product of Eqs. (29) and (27) as follows:

$$D_0 = \psi D^* = 0.51134296 \text{ m} \quad (26)$$

A comparison between the exact pipe diameter obtained from Eq. (33) and the approximate diameter predicted by Eq. (26) reveals an exceptionally small relative deviation of only about 0.023%, despite the complete absence of any iterative correction. This remarkable level of accuracy demonstrates the high predictive capability of the proposed analytical approximation. The approximate diameter is subsequently employed to evaluate the initial estimate z_0 through Eq. (15a), thereby providing a physically consistent and highly accurate starting point for Halley's iterative procedure.

According to Eqs. (15a) and (26), the initial guess is as follows:

$$z_0 = \frac{1}{\sqrt{D_0}} \approx 1.398440068 \text{ m}^{-1/2} \quad (15a) \text{The computation allowed us to compute the follow}$$

$$F(z_0) = \alpha z_0^5 + 2 \ln(z_0) + \ln(\beta + \gamma z_0) \approx 0.00459685 \quad (21a)$$

$$F'(z) = 5\alpha z_0^4 + \frac{2}{z_0} + \frac{\gamma}{\beta + \gamma z_0} \approx 39.0053432 \quad (22)$$

and

$$F''(z_0) = 20\alpha z_0^3 - \frac{2}{z_0^2} - \frac{\gamma^2}{(\beta + \gamma z_0)^2} \approx 73.366003 \quad (25)$$

Using the analytical initial estimate together with the quantities evaluated from Eqs. (15a), (21a), (22), and (25), the application of a single Halley iteration through Eq. (24) yields the following refined estimate of the transformed variable:

$$z_1 = 1.3983222 \text{ m}^{-1/2} \quad (24)$$

Finally, the required pipe diameter is readily obtained by substituting the computed value of z_1 into Eq. (15), yielding the following result:

$$D = \frac{1}{z_1^2} \approx 0.51142917 \text{ m} \quad (15)$$

Remarkably, the numerical results demonstrate the outstanding effectiveness of the Rough Model Method (RMM) in providing an initial approximation of the transformed variable. Starting from the analytical estimate delivered by the RMM, a single Halley iteration is sufficient to recover a pipe diameter of $D = 0.51142917 \text{ m}$, whereas the exact solution of the transformed equation is $D = 0.51146176 \text{ m}$. The corresponding relative deviation is only 0.00637%, which is negligible for virtually all engineering applications.

This result is particularly significant because it reveals that the RMM initial estimate already lies in the immediate neighborhood of the exact solution. Consequently, Halley's third-order correction no longer serves as a conventional iterative solver, but rather as a final refinement step that removes the residual approximation error. The overall computational procedure is therefore reduced, in practice, to a single correction iteration, combining the analytical simplicity of the Rough Model Method with the rapid local convergence of Halley's algorithm.

From a computational perspective, this finding represents a substantial improvement over traditional iterative procedure. Instead of performing repeated coupled iterations on the Darcy-Weisbach and Colebrook-White equations, the proposed methodology provides an almost explicit solution in which the analytical approximation supplied by the RMM captures nearly the entire nonlinear behavior of the problem, leaving only a minute residual error to be corrected numerically. The resulting algorithm is therefore

exceptionally simple, computationally efficient, and highly accurate, making it particularly attractive for engineering design calculations as well as for implementation in hydraulic software.

To further confirm the validity and practical robustness of the proposed procedure, a second numerical example is considered below.

2. Let's consider the following data:

$$Q = 0.40 \text{ m}^3/\text{s}, S_f = 0.00006, \varepsilon = 0.0008 \text{ m}, \nu = 10^{-6} \text{ m}^2/\text{s}, g = 9.81 \text{ m/s}^2.$$

2.1 Solution using Halley's method [Eq. (24)]

Using Eqs. (16), (17), and (18) the following constants are obtained:

$$\alpha \approx 5.4041908310, \beta \approx 0.0002162162, \text{ and } \gamma \approx 0.000023139 \quad (34)$$

Substituting these constants into the governing Eq. (21a), the governing equation is obtained as follows:

$$F(z) = 5.4041908310 z^5 + 2 \ln(z) + \ln(0.0002162162 + 0.000023139 z)$$

The exact physically admissible positive root is obtained through the application of Halley's iterative method expressed by Eq. (24); the final result is as follows:

$$z = 1.0860186965 \text{ m}^{-1/2} \quad (35)$$

The sought diameter D is given by Eq. (15) as follows:

$$D = \frac{1}{z^2} \approx 0.84786244 \text{ m} \quad (15)$$

2.2 Approximate solution using the initial guess [Eq. (15a)]

To demonstrate the general applicability of the proposed methodology, the computational procedure established in subsection 1.2 is repeated herein using a different set of hydraulic parameters.

The rough pipe diameter is computed using Eq. (29) as follows:

$$D^* = \left(\frac{Q^2}{2\pi^2 g S_f} \right)^{1/5} = 1.06609031 \text{ m} \quad (29)$$

Thus, Eq. (6a) allows computing the Reynolds number R^* in the rough model as follows:

$$R^* = \frac{4Q}{\pi D^* \nu} = 477,722.959 \quad (6a)$$

Then, Eq. (27) allow computing the dimensionless linear dimensions correction factor ψ as follows:

$$\psi = 1.35 \left[-\log \left(\frac{\varepsilon / D^*}{4.75} + \frac{8.5}{R^*} \right) \right]^{-2/5} \approx 0.79522049 \quad (27)$$

According to Eq. (26), the approximate sought diameter D_0 is given by the product of Eqs. (29) and (27) as follows:

$$D_0 = \psi D^* = 0.84777686m \quad (26)$$

The comparison between the approximate diameter D_0 and the exact diameter D obtained in Subsection 2.1 reveals an exceptionally small relative deviation of only about 0.01%, thereby confirming the remarkable accuracy of the proposed analytical approximation based on the Rough Model Method (RMM).

As in the previous numerical example, the present application further confirms that the approximate diameter D_0 , obtained without any iterative computation, is remarkably close to the exact solution. The resulting relative deviations are extremely small and are well within the accuracy requirements of the vast majority of practical hydraulic engineering applications. Consequently, the proposed analytical approximation may, in many cases, be used directly as a reliable design tool, thereby completely eliminating the need for iterative calculations.

Nevertheless, certain engineering problems may require substantially higher numerical accuracy. In such cases, it is worth recalling that the initial estimate z_0 , which is related to the approximate diameter D_0 through Eq. (15a), serves as the starting value for Halley's iterative model represented by Eq. (24). Owing to the exceptional quality of this analytically derived initial estimate, only a single Halley correction is generally sufficient to recover the exact solution with virtually machine-precision accuracy. The implementation of this final refinement step is presented in the following subsection.

According to Eqs. (15a) and (26), the initial guess is as follows:

$$z_0 = \frac{1}{\sqrt{D_0}} \approx 1.086073511 \text{ m}^{-1/2} \quad (15a) \text{ This value allows us to compute the following:}$$

$$F(z_0) = \alpha z_0^5 + 2 \ln(z_0) + \ln(\beta + \gamma z_0) \approx 0.00216677 \quad (21a)$$

$$F'(z) = 5\alpha z_0^4 + \frac{2}{z_0} + \frac{\gamma}{\beta + \gamma z_0} \approx 42.7689881 \quad (22)$$

and

$$F''(z_0) = 20\alpha z_0^3 - \frac{2}{z_0^2} - \frac{\gamma^2}{(\beta + \gamma z_0)^2} \approx 136.613782 \quad (25)$$

Using the quantities previously computed from Eqs. (15a), (21a), (22), and (25), a single application of Halley's iterative scheme, Eq. (24), yields the refined estimate z_1 . The corresponding result is

$$z_1 = 1.08602284 \text{ m}^{-1/2} \quad (24)$$

Furthermore, the sought pipe diameter is directly obtained by substituting the computed value of z_1 into Eq. (15), resulting in the following result:

$$D = \frac{1}{z_1^2} \approx 0.847855962 \text{ m} \quad (15)$$

The comparison with the exact diameter obtained in subsection 2.1 reveals that the diameter resulting from a single Halley refinement differs by only about 0.000763%. Such an exceedingly small relative deviation demonstrates the outstanding effectiveness of the proposed computational procedure. It confirms that the analytical initial estimate is already located in the immediate vicinity of the exact solution, so that a single Halley correction is sufficient to reduce the residual error to a practically negligible level. Consequently, the proposed methodology combines the simplicity of an analytical approximation with the accuracy of an exact iterative solution, while requiring only one refinement step for engineering applications demanding exceptionally high numerical precision.

From a practical engineering standpoint, it should be emphasized that determining the pipe diameter with machine-precision accuracy is seldom a design requirement. The diameter obtained from the hydraulic calculations represents a theoretical value that is not generally available as a commercial product. In engineering practice, the calculated diameter is used solely as a basis for selecting the nearest standardized pipe size, which is conventionally chosen as the first commercially available diameter equal to or greater than the theoretical value.

Consequently, the minute differences observed between the approximate and the exact solutions are, in most practical situations, of no engineering significance. Relative deviations of only a few hundredths, or even a few thousandths, of a percent have virtually no influence on the final design, since the adopted pipe diameter is ultimately dictated by standard manufacturing dimensions rather than by the exact numerical value predicted by the mathematical model.

This practical consideration further highlights the relevance of the proposed analytical approximation. Even without iterative refinement, the predicted diameter is sufficiently accurate to ensure the selection of the same standardized pipe size as that obtained from

the exact solution. When exceptionally high numerical accuracy is required for research or verification purposes, the proposed Halley correction provides an efficient means of recovering the exact solution. However, for routine hydraulic design, the analytical approximation alone is generally more than adequate, combining simplicity, computational efficiency, and engineering reliability.

ASYMPTOTIC SOLUTIONS

Although the transformed governing equation developed in the preceding sections is valid throughout the entire turbulent-flow regime, its mathematical structure becomes considerably simpler in the two fundamental asymptotic limits corresponding to hydraulically smooth and fully rough flow conditions. These limiting regimes are of particular importance because they represent the two extreme resistance mechanisms governing turbulent flow in circular pressurized pipes.

The purpose of the present section is to investigate the asymptotic behavior of the proposed formulation under these limiting conditions and to derive the corresponding analytical expressions for the pipe diameter. Unlike the general formulation, which remains transcendental and requires the determination of the unique physically admissible root of the transformed equation, the asymptotic equations belong to a class of nonlinear transcendental equations for which exact closed-form solutions can be established through the theory of special functions.

The resulting analytical developments demonstrate that the pipe diameter can be expressed in closed form for both limiting regimes without introducing empirical approximations or numerical simplifications. These exact asymptotic solutions are obtained through rigorous mathematical transformations while fully preserving the physical assumptions characterizing each hydraulic regime.

Beyond their theoretical significance, these asymptotic solutions provide valuable insight into the respective influence of wall roughness and viscous effects on pipe sizing. They also establish a direct analytical connection between the governing hydraulic equations and exact closed-form mathematical representations, thereby extending the applicability of the proposed framework beyond the general numerical solution.

The asymptotic analysis presented herein therefore constitutes a natural complement to the general formulation developed previously. Together, the general transformed equation and its exact asymptotic solutions provide a comprehensive analytical description of pipe-diameter determination over the entire turbulent-flow domain.

Fully rough pipes

In the fully rough limit, the term βz^2 dominates the logarithmic argument, so Eq. (19) becomes as follows:

$$\alpha z^5 + \ln(\beta z^2) = 0 \quad (36)$$

Expanding the logarithm yields the following:

$$\alpha z^5 + \ln(\beta) + 2 \ln(z) = 0 \quad (36a)$$

Let's adopt the following change in variables:

$$y = z^5 \quad (37)$$

Then, the following can be written:

$$\ln(z) = \frac{1}{5} \ln(y) \quad (38)$$

Substituting Eq. (38) into Eq. (36a) results in the following:

$$\alpha y + \ln(\beta) + \frac{2}{5} \ln(y) = 0 \quad (39)$$

Eq. (39) is rearranged in the following form:

$$\frac{5\alpha}{2} y + \ln(y) = -\frac{5}{2} \ln(\beta) \quad (40)$$

Exponentiating both sides, and after rearrangement, Eq. (40) becomes as follows:

$$y \exp\left(\frac{5\alpha}{2} y\right) = \beta^{-5/2} \quad (41)$$

Eq. (41) can be rewritten as follows:

$$\frac{5\alpha}{2} y \exp\left(\frac{5\alpha}{2} y\right) = \frac{5\alpha}{2} \beta^{-5/2} \quad (42)$$

By definition of the Lambert function, the following can be written:

$$\frac{5\alpha}{2} y = W_0\left(\frac{5\alpha}{2} \beta^{-5/2}\right) = W_0(A) \quad (43)$$

Where the argument A is given as follows:

$$A = \frac{5\alpha}{2} \beta^{-5/2} \quad (44)$$

Therefore, the following result can be written:

$$y = \frac{2}{5\alpha} W_0 \left(\frac{5\alpha}{2} \beta^{-5/2} \right)$$

Therefore, according to Eq. (37), the following can be written:

$$z = \left[\frac{2}{5\alpha} W_0 \left(\frac{5\alpha}{2} \beta^{-5/2} \right) \right]^{1/5} \quad (45)$$

Using the change of variables expressed in Eq. (15), the sought full rough pipe diameter is given as follows:

$$D_R = \left[\frac{2}{5\alpha} W_0 \left(\frac{5\alpha}{2} \beta^{-5/2} \right) \right]^{-2/5} \quad (46)$$

The subscript « R » denotes « Rough ».

Eq. (46) reveals another fundamental property of the proposed asymptotic formulation. Since the parameters α and β depend respectively on the discharge Q , the friction slope S_f , the absolute roughness ε , and the gravitational acceleration g , the corresponding asymptotic diameter D_R is governed exclusively by these quantities. Remarkably, neither the Reynolds number nor the kinematic viscosity ν appears in the final analytical expression.

This result constitutes a direct mathematical confirmation of the physical characteristics of the fully rough turbulent regime. Under these flow conditions, the viscous sublayer becomes negligible compared with the roughness height, and the wall resistance is controlled entirely by the surface roughness. Consequently, viscous effects no longer influence the hydraulic behavior of the flow, and the required pipe diameter becomes completely independent of both the Reynolds number and the fluid viscosity.

The complete disappearance of the viscous parameters from the final solution is therefore far from being a mere algebraic consequence. It demonstrates that the proposed analytical derivation faithfully reproduces the fundamental physics of fully rough turbulent flow. The asymptotic solution naturally recovers the well-established hydraulic principle that, in the fully rough regime, the flow resistance depends solely on wall roughness and the imposed hydraulic conditions, while viscous effects become entirely irrelevant (Achour and Amara, 2020b; 2020c; Achour and Amara, 2021a).

This remarkable agreement between the analytical formulation and the physical behavior of turbulent flow provides a further validation of the proposed framework. It confirms that the asymptotic solution is not only mathematically rigorous but also fully consistent with the classical theory of turbulent pipe flow, thereby reinforcing both the physical significance and the general applicability of the present analytical approach.

Hydraulically smooth pipes

In the hydraulically smooth limit, the term γz^3 dominates the logarithmic argument, so the governing Eq. (19) becomes as follows:

$$\alpha z^5 + \ln(\gamma z^3) = 0 \quad (47)$$

Expanding the logarithm yields the following:

$$\alpha z^5 + \ln(\gamma) + 3\ln(z) = 0 \quad (48)$$

Substituting Eqs. (37) and (38) into Eq. (48) results in what follows:

$$\alpha y + \ln(\gamma) + \frac{3}{5}\ln(y) = 0 \quad (49)$$

Eq. (49) can be written in the following form:

$$\frac{5\alpha}{3}y + \ln(y) = -\frac{5}{3}\ln(\gamma) \quad (50)$$

Exponentiating both sides, and after rearrangement, Eq. (50) is written as follows:

$$y \exp\left(\frac{5\alpha}{3}y\right) = \gamma^{-5/3} \quad (51)$$

Eq. (51) can be also rewritten as follows:

$$\frac{5\alpha}{3}y \exp\left(\frac{5\alpha}{3}y\right) = \frac{5\alpha}{3}\gamma^{-5/3} \quad (52)$$

By definition of the Lambert function, Eq. (52) allows us writing what follows:

$$\frac{5\alpha}{3}y = W_0\left(\frac{5\alpha}{3}\gamma^{-5/3}\right) = W_0(A) \quad (53)$$

Where the argument A is given as follows:

$$A = \frac{5\alpha}{3}\gamma^{-5/3} \quad (54)$$

Therefore, the following result can be written:

$$y = \frac{3}{5\alpha}W_0\left(\frac{5\alpha}{3}\gamma^{-5/3}\right) \quad (55)$$

Therefore, according to Eq. (37), Eq. (55) becomes as follows:

$$z = \left[\frac{3}{5\alpha} W_0 \left(\frac{5\alpha}{3} \gamma^{-5/3} \right) \right]^{1/5} \quad (56)$$

According to Eq. (15), the sought hydraulically smooth pipe diameter is given as follows:

$$D_S = \left[\frac{3}{5\alpha} W_0 \left(\frac{5\alpha}{3} \gamma^{-5/3} \right) \right]^{-2/5} \quad (57)$$

The subscript « S » denotes « Smooth ».

An important feature of the asymptotic solution given by Eq. (57) deserves particular attention. Since the parameter β depends exclusively on the discharge Q , the friction slope S_f , the kinematic viscosity ν , and the gravitational acceleration g , the corresponding asymptotic diameter D_s is entirely determined by these hydraulic quantities. Remarkably, the absolute roughness ε does not appear in the final analytical expression.

This result is not merely a mathematical consequence of the derivation; it constitutes a direct confirmation of the physical nature of the hydraulically smooth turbulent regime. Under these flow conditions, the wall roughness becomes hydraulically ineffective, and the flow resistance is governed solely by viscous effects. Consequently, the required pipe diameter is completely independent of the roughness characteristics of the conduit and is controlled only by the imposed hydraulic conditions and the fluid properties.

The disappearance of the roughness parameter from the final solution therefore provides a strong physical validation of the proposed analytical formulation. It demonstrates that the asymptotic solution naturally recovers the fundamental behavior expected for hydraulically smooth turbulent flow without the need for additional assumptions or empirical corrections (Achour and Amara, 2021b). This consistency between the analytical derivation and the underlying hydraulic physics further reinforces the soundness and generality of the proposed framework.

Numerical examples

1. Fully rough pipe

Consider the following data:

$$Q = 0.25 \text{ m}^3/\text{s}, S_f = 0.0006, \varepsilon = 0.001 \text{ m}, g = 9.81 \text{ m/s}^2.$$

According to Eq. (16), the following result is obtained:

$$\alpha = \frac{\sqrt{2} Q \ln(10)}{\pi \sqrt{g S_f}} \approx 3.37761927 \quad (16)$$

Eq. (17) gives what follows:

$$\beta = \frac{\varepsilon}{3.7} \approx 0.00027027 \quad (17)$$

Therefore, the argument A is given by Eq. (44) as follows:

$$A = \frac{5\alpha}{2} \beta^{-5/2} = 7031613843.1878200 \quad (44)$$

The iterative process applied in the Halley's method provides the following result:

$$W_0\left(\frac{5\alpha}{2} \beta^{-5/2}\right) = W_0(7031613843.1878200) = 19.693398598534$$

Thus, the sought fully rough pipe diameter is obtained as follows:

$$D_R = \left[\frac{2}{5\alpha} W_0\left(\frac{5\alpha}{2} \beta^{-5/2}\right) \right]^{-2/5} \approx 0.7126758 \text{ m} \quad (46)$$

The principal branch of the Lambert W function, $W_0(A)$, was not obtained from a closed-form analytical expression, since no such expression exists. Instead, it was determined numerically by exploiting the defining implicit relation of the Lambert function, expressed as follows:

$$W(A) \exp[W(A)] = A \quad (58)$$

The computation begins with an asymptotic approximation, which provides an accurate initial estimate for sufficiently large positive values of (A) . This preliminary estimate is subsequently refined through a high-order iterative scheme, namely Halley's method. At each iteration, the current approximation is corrected using the nonlinear function $f(W) = W \exp(W)$ together with its first and second derivatives, yielding rapid cubic convergence toward the exact solution.

For the specific argument $A = 7031613843.1878200$, the iterative process converges after only a few iterations to the unique solution belonging to the principal branch, giving
 $W_0(7031613843.1878200) \approx 19.6933985985336$.

The accuracy of this result is confirmed by direct substitution into the defining Eq. (58), demonstrating that it reproduces the original argument to machine precision. This numerical strategy is the standard approach adopted in modern scientific computing libraries due to its robustness, rapid convergence, and near machine-precision accuracy.

However, the authors recommend using the approximate W_0 expressed as follows:

$$W_0(A) \approx L_1 - L_2 + \frac{L_2}{L_1} + \frac{L_2(L_2 - 2)}{2L_1^2} + \frac{L_2(2L_2^2 - 9L_2 + 6)}{6L_1^3} \tag{59}$$

where

$$L_1 = \ln(z) \tag{60}$$

and

$$L_2 = \ln[\ln(z)] = \ln(L_1) \tag{61}$$

The derived authors' approximation (59) provides an excellent starting value for positive arguments and was subsequently refined using Halley's third-order iteration. In the present case, Eq. (59), along with Eqs. (60) and (61), gives the following result:

$$W_0(A)_{\text{approximate}} = 19.693422032070 \tag{62}$$

Thus, the deviation between the exact and approximate value of W_0 is about only 0.000119%. Consequently, using the approximate value given in Eq. (62), the approximate rough pipe diameter is as follows:

$$D_R(\text{approximate}) = \left[\frac{2}{5\alpha} W_0(\text{approximate}) \right]^{-2/5} \approx 0.71267546 \text{ m} \tag{63}$$

Therefore, a comparison between the exact diameter obtained from Eq. (46) and the approximate diameter predicted by Eq. (63) reveals an exceptionally small relative deviation of only 0.0000476%. This remarkably small discrepancy demonstrates the outstanding accuracy of the proposed approximation Eq. (59). Such a negligible error indicates that Eq. (63) reproduces the exact solution with near-perfect precision, making it highly reliable for practical engineering and scientific applications. The result also confirms the validity of the underlying approximation and shows that it can be confidently used in place of the exact expression without iterative calculations or significant loss of accuracy.

2. Hydraulically smooth pipe

Let's consider the following data to compute the hydraulically smooth pipe diameter expressed by the analytical Eq. (57).

$$Q = 0.35 \text{ m}^3/\text{s}, S_f = 0.0003, \nu = 0.000001 \text{ m}^2/\text{s}, g = 9.81 \text{ m/s}^2.$$

According to Eqs. (16) and (18), the following results are obtained:

$$\alpha = \frac{\sqrt{2} Q \ln(10)}{\pi \sqrt{g S_f}} \approx 6.68734497 \quad (16)$$

$$\gamma = \frac{2.51 \nu}{\sqrt{2 g S_f}} = 0.0000327163 \quad (18)$$

Therefore, the argument A is given by Eq. (54) as follows:

$$A = \frac{5\alpha}{3} \gamma^{-5/3} = 333040069.9446820 \quad (54)$$

Substituting Eqs. (16) (18) and (54) into Eq. (57), the sought hydraulically smooth pipe diameter is obtained as follows:

$$D_S = \left[\frac{3}{5\alpha} W_0 \left(\frac{5\alpha}{3} \gamma^{-5/3} \right) \right]^{-2/5} \approx 0.84858219 \text{ m} \quad (64)$$

To further validate the proposed approximation, we examine whether the diameter obtained from the exact formulation can be accurately recovered using the approximate Eq. (57) in combination with the approximate Eq. (59).

In the present case, approximate Eq. (59), along with Eqs. (60) and (61), gives the following result:

$$W_0(A)_{\text{approximate}} = 16.80229582 \quad (65)$$

The deviation between the exact and approximate value of W_0 is about only 0.000213%. Consequently, using Eq. (64), along with the approximate value given in Eq. (65), the approximate hydraulically smooth pipe diameter is as follows:

$$D_S(\text{approximate}) = \left[\frac{3}{5\alpha} W_0(\text{approximate}) \right]^{-2/5} \approx 0.84858146 \text{ m} \quad (66)$$

A comparison between the exact diameter calculated from Eq. (64) and the approximate value obtained using Eq. (66) yields an extremely small relative deviation of only 0.0000853%. This exceptional level of agreement highlights the remarkable accuracy of

the proposed approximation given by Eq. (59). The virtually negligible error demonstrates that Eq. (66) provides an almost exact reproduction of the reference solution, thereby ensuring an excellent degree of reliability for both engineering practice and scientific analysis. Furthermore, these findings validate the underlying approximation and establish that the explicit expression can confidently replace the exact formulation, eliminating the need for iterative computations while preserving virtually complete numerical accuracy.

SERIES-BASED TREATMENT OF THE GENERAL TRANSFORMED COLEBROOK-WHITE DIAMETER EQUATION

Objective

Although the general transformed Colebrook-White diameter equation, Eq. (21a), does not appear to admit a closed-form solution in terms of classical elementary or Lambert-type functions, it can be treated rigorously through asymptotic series representations. In this approach, the exact Lambert W -based solutions corresponding to the fully rough and hydraulically smooth limits serve as analytical kernels, while the neglected resistance contribution is introduced through a convergent logarithmic expansion. This provides a systematic bridge between the two limiting regimes and offers a mathematically controlled approximation of the full solution.

The objective of the present section is to establish a unified analytical framework capable of extending the exact asymptotic solutions toward the general turbulent-flow regime. Rather than seeking an alternative numerical procedure, the proposed development aims at revealing the mathematical structure of the transformed Colebrook-White equation and at demonstrating how the complete solution may be represented as a sequence of successive analytical corrections applied to the exact asymptotic kernels. Consequently, the fully rough and hydraulically smooth solutions are no longer viewed as isolated limiting cases, but rather as the leading-order terms of two complementary asymptotic expansions converging toward the general solution.

This analytical strategy provides a rigorous interpretation of the complete transformed equation by explicitly identifying the mathematical origin of the residual terms neglected in each asymptotic approximation. Each successive correction possesses a clear physical significance, accounting for the gradual reintroduction of the resistance mechanism omitted in the corresponding limiting regime. As a result, the proposed methodology establishes a direct correspondence between the asymptotic physics of turbulent flow and the mathematical structure of the transformed equation.

Beyond its theoretical interest, the proposed series-based formulation offers a systematic procedure for progressively improving the asymptotic solutions while preserving their exact analytical character. It therefore constitutes a natural extension of the Lambert W -based formulations derived previously and provides further insight into the mathematical properties of the general transformed Colebrook-White diameter equation.

Formulation of the hydraulic design equations

Let's recall the transformed Colebrook-White diameter equation as follows:

$$F(z) = \alpha z^5 + \ln(\beta z^2 + \gamma z^3) = 0 \quad (21)$$

Eq. (21) has also been written in the following form:

$$F(z) = \alpha z^5 + 2\ln(z) + \ln(\beta + \gamma z) \quad (21a)$$

The difficulty comes from the following term,

$$\ln(\beta + \gamma z) \quad (67)$$

because it contains the simultaneous contribution of roughness and viscosity. Therefore, the general equation cannot be reduced directly to a single Lambert W -type form. However, it can be treated analytically by expanding the logarithmic term around the two limiting hydraulic regimes.

Fully rough expansion

In the fully rough regime, the roughness contribution dominates, Eq. (21) allows us to write the following:

$$\beta z^2 \gg \gamma z^3 \quad (68)$$

This is equivalent to what follows:

$$\frac{\gamma z}{\beta} \ll 1 \quad (69)$$

Eq. (69) possesses a clear physical interpretation that extends well beyond its mathematical role in establishing the convergence of the logarithmic expansion. Indeed, the dimensionless ratio, expressed by Eq. (69), measures the relative magnitude of the viscous contribution with respect to the roughness contribution appearing in the transformed Colebrook-White equation [Eq. (21)]. Consequently, it may be regarded as a direct indicator of the competition between the two fundamental mechanisms governing turbulent flow resistance.

When the inequality (69) is fully satisfied, the viscous contribution becomes negligible compared with the roughness term. Under these conditions, the wall roughness controls the hydraulic resistance, while viscous effects exert only a minor influence on the flow. This is precisely the physical definition of the fully rough turbulent regime. The logarithmic series, that will be introduced subsequently, therefore possesses not only a rigorous mathematical justification but also a direct hydraulic interpretation, since each

successive term represents an increasingly smaller viscous correction to the dominant roughness-controlled solution.

It is worth emphasizing that the ratio defined by Eq. (6) is entirely dimensionless and naturally emerges from the analytical reformulation of the governing equations. Unlike empirical transition criteria, it is derived directly from the mathematical structure of the transformed Colebrook-White equation itself. Consequently, Eq. (6) provides both the criterion governing the convergence of the asymptotic expansion and a quantitative measure of the relative importance of viscous effects in fully rough turbulent flow.

On the other hand, Eq. (21) can be written in the following form:

$$F(z) = \alpha z^5 + \ln(\beta z^2) + \ln\left(1 + \frac{\gamma z}{\beta}\right) = 0 \quad (70)$$

Since the inequality expressed by (69) is satisfied, the logarithm expansion gives the following:

$$\ln\left(1 + \frac{\gamma z}{\beta}\right) = \frac{\gamma z}{\beta} - \frac{1}{2}\left(\frac{\gamma z}{\beta}\right)^2 + \frac{1}{3}\left(\frac{\gamma z}{\beta}\right)^3 - \frac{1}{4}\left(\frac{\gamma z}{\beta}\right)^4 + \dots \quad (71)$$

Substituting Eq. (71) into Eq. (70) yields the following:

$$\alpha z^5 + \ln(\beta z^2) + \frac{\gamma z}{\beta} - \frac{1}{2}\left(\frac{\gamma z}{\beta}\right)^2 + \frac{1}{3}\left(\frac{\gamma z}{\beta}\right)^3 - \frac{1}{4}\left(\frac{\gamma z}{\beta}\right)^4 = 0 \quad (72)$$

If only the dominant term is retained, one obtains what follows:

$$\alpha z^5 + \ln(\beta z^2) = 0 \quad (73)$$

This is the fully rough asymptotic equation. Its solution is given by Eq. (45) as follows:

$$z_R = \left[\frac{2}{5\alpha} W_0 \left(\frac{5\alpha}{2} \beta^{-5/2} \right) \right]^{1/5} \quad (45)$$

Eq. (72) gives the leading-order fully rough value. The remaining terms in Eq. (72) then provide successive corrections accounting for the viscous contribution that was neglected in the pure fully rough limit.

Thus, the full solution may be represented as follows:

$$z = z_R + \Delta_R \quad (74)$$

where Δ_R is a correction induced by the additional logarithmic series in Eq. (72). A first-order correction may be estimated by writing what follows:

$$F(z_R + \Delta_R) \approx F(z_R) + F'(z_R)\Delta_R \quad (75)$$

Since z_R satisfies Eq. (73), the residual of the full equation at $z = z_R$ is given as follows, according to Eqs. (70) and (73):

$$F(z_R) = \ln\left(1 + \frac{\gamma z_R}{\beta}\right) \quad (76)$$

Therefore, Eqs. (75) and (76) allow us to write the following:

$$\Delta_R \approx -\frac{\ln\left(1 + \frac{\gamma z_R}{\beta}\right)}{F'(z_R)} \quad (77)$$

The derivative of the full function, Eq. (70), for $z = z_R$, is given as follows:

$$F'(z_R) = 5\alpha z_R^4 + \frac{2}{z_R} + \frac{\gamma}{\beta + \gamma z_R} \quad (78)$$

Substituting Eq. (78) into Eq. (77) yields the following results:

$$\Delta_R \approx -\frac{\ln\left(1 + \frac{\gamma z_R}{\beta}\right)}{5\alpha z_R^4 + \frac{2}{z_R} + \frac{\gamma}{\beta + \gamma z_R}} \quad (79)$$

The corrected fully rough expansion is therefore as follows:

$$z \approx z_R - \frac{\ln\left(1 + \frac{\gamma z_R}{\beta}\right)}{5\alpha z_R^4 + \frac{2}{z_R} + \frac{\gamma}{\beta + \gamma z_R}} \quad (80)$$

Using Eq. (15), the sought diameter D is then given as follows:

$$D \approx \left[z_R - \frac{\ln\left(1 + \frac{\gamma z_R}{\beta}\right)}{5\alpha z_R^4 + \frac{2}{z_R} + \frac{\gamma}{\beta + \gamma z_R}} \right]^{-2} \quad (81)$$

Eq. (79) provides the first-order residual correction required to transform the fully rough asymptotic solution into an approximation of the complete transformed Colebrook-White equation. From a mathematical viewpoint, Δ_R represents the response of the exact nonlinear model to the omission of the viscous contribution in the leading-order asymptotic formulation. It therefore quantifies the discrepancy between the fully rough approximation and the complete turbulent-flow model.

An important feature of Eq. (79) is that the residual correction is always negative. This property follows directly from the mathematical structure of the transformed equation. However, the negative sign has a clear physical interpretation. The fully rough asymptotic solution is obtained by neglecting the viscous contribution to the flow resistance. As a result, the governing equation slightly overestimates the value of the transformed variable z . The residual correction therefore acts in the opposite direction, reducing z_R in order to restore the exact balance between the roughness-controlled and viscosity-controlled resistance mechanisms. The correction is thus not arbitrary; it represents the analytical compensation required by the reintroduction of viscous effects into the complete model.

Another remarkable feature of Eq. (79) is that the magnitude of the correction is governed by the following logarithmic quantity:

$$\ln\left(1 + \frac{\gamma z_R}{\beta}\right) \tag{82}$$

which is precisely the term neglected during the asymptotic simplification. Consequently, the correction automatically tends to zero, written as follows:

$$\frac{\gamma z_R}{\beta} \rightarrow 0 \tag{83}$$

that is, when the flow approaches the ideal fully rough limit. Under these conditions, the following can be written:

$$\Delta_R \rightarrow 0 \tag{84}$$

and the asymptotic solution becomes identical to the exact solution of the complete equation. This behavior provides a strong mathematical validation of the proposed asymptotic formulation and demonstrates its complete consistency with the limiting physics of fully rough turbulent flow.

As a concluding sentence, Eq. (79) demonstrates that the residual correction is by no means an empirical adjustment. Rather, it constitutes the leading-order analytical contribution to the error generated by suppressing the viscous term in the asymptotic formulation. Consequently, the residual Δ_R acquires a clear variational interpretation: it measures the sensitivity of the fully rough asymptotic solution to the gradual restoration of viscous effects. This perspective elevates the correction beyond a purely numerical refinement and establishes it as an intrinsic analytical component of the proposed

formulation, thereby reinforcing both its mathematical rigor and its physical significance.

Hydraulically smooth expansion

In the hydraulically smooth regime, the viscous contribution dominates, so that writing the following inequality:

$$\gamma z^3 \gg \beta z^2 \quad (85)$$

This is equivalent to what follows:

$$\frac{\beta}{\gamma z} \ll 1 \quad (86)$$

A similar interpretation applies to the dimensionless ratio introduced in Eq. (86). This quantity represents the converse situation, namely the relative importance of wall roughness compared with viscous effects. When the inequality (86) is satisfied, the influence of surface roughness becomes negligible, whereas viscous effects entirely govern the hydraulic resistance. The flow therefore behaves as hydraulically smooth, despite the presence of a finite physical roughness on the pipe wall.

From a mathematical viewpoint, inequality (86) constitutes the convergence criterion for the logarithmic expansion employed in the derivation of the smooth-flow asymptotic solution. From a physical viewpoint, however, it carries a much broader significance. It quantifies the progressive disappearance of roughness effects as the viscous sublayer becomes sufficiently thick to completely mask the wall irregularities. Consequently, the successive terms of the logarithmic series may be interpreted as progressively smaller roughness corrections applied to the dominant viscous solution.

An important consequence of this observation is that Eqs. (69) and (86) form a complementary pair of dimensionless parameters describing the same physical phenomenon from opposite perspectives. The former measures the residual influence of viscosity in predominantly rough turbulent flow, whereas the latter measures the residual influence of roughness in predominantly smooth turbulent flow. Together, they provide a unified interpretation of the transition between the two asymptotic regimes and establish a direct correspondence between the mathematical structure of the series expansions and the underlying physics of turbulent wall resistance.

On the other hand, the governing Eq. (21) can be written as follows:

$$F(z) = \alpha z^5 + \ln(\gamma z^3) + \ln\left(1 + \frac{\beta}{\gamma z}\right) = 0 \quad (87)$$

Using the logarithm expansion, Eq. (87) can be written in the following form:

$$\alpha z^5 + \ln\left(\gamma z^3\right) + \frac{\beta}{\gamma z} - \frac{1}{2}\left(\frac{\beta}{\gamma z}\right)^2 + \frac{1}{3}\left(\frac{\beta}{\gamma z}\right)^3 - \frac{1}{4}\left(\frac{\beta}{\gamma z}\right)^4 = 0 \quad (88)$$

Keeping only the dominant terms gives the following:

$$\alpha z^5 + \ln\left(\gamma z^3\right) = 0 \quad (89)$$

This is the hydraulically smooth asymptotic equation. Its solution is given by Eq. (56), recalled as follows:

$$z_S = \left[\frac{3}{5\alpha} W_0 \left(\frac{5\alpha}{3} \gamma^{-5/3} \right) \right]^{1/5} \quad (56)$$

The complete solution may then be written as follows:

$$z = z_S + \Delta_S \quad (90)$$

Using a first-order correction, one may write the following:

$$F(z_S + \Delta_S) \approx F(z_S) + F'(z_S)\Delta_S \quad (91)$$

Since z_S satisfies Eq. (89), the residual of the full equation at $z = z_S$ is given as follows, according to Eqs. (87) and (91):

$$F(z_S) = \ln\left(1 + \frac{\beta}{\gamma z}\right) \quad (92)$$

Therefore, Eqs. (91) and (92) allow us to write the following:

$$\Delta_S \approx - \frac{\ln\left(1 + \frac{\beta}{\gamma z}\right)}{F'(z_S)} \quad (93)$$

The derivative of the full function, Eq. (87), for $z = z_S$, is given as follows:

$$F'(z_S) = 5\alpha z_S^4 + \frac{2}{z_S} + \frac{\gamma}{\beta + \gamma z_S} \quad (94)$$

Substituting Eq. (94) into Eq. (93) yields the following results:

$$\Delta_S \approx - \frac{\ln\left(1 + \frac{\beta}{\gamma z}\right)}{5\alpha z_S^4 + \frac{2}{z_S} + \frac{\gamma}{\beta + \gamma z_S}} \quad (95)$$

The interpretation and conclusions drawn from Eq. (79) remain fully applicable to Eq. (95), and are therefore not repeated for the sake of brevity.

The corrected hydraulically smooth expansion is therefore as follows:

$$z \approx z_S - \frac{\ln\left(1 + \frac{\beta}{\gamma z}\right)}{5\alpha z_S^4 + \frac{2}{z_S} + \frac{\gamma}{\beta + \gamma z_S}} \quad (96)$$

Using Eq. (15), the sought diameter D is then given as follows:

$$D \approx \left[z_S - \frac{\ln\left(1 + \frac{\beta}{\gamma z_S}\right)}{5\alpha z_S^4 + \frac{2}{z_S} + \frac{\gamma}{\beta + \gamma z_S}} \right]^{-2} \quad (97)$$

Numerical examples

Example 1

Let's consider the following data:

$Q = 0.30 \text{ m}^3/\text{s}$, $S_f = 0.005$, $\varepsilon = 0.001 \text{ m}$, $\nu = 0.000001 \text{ m}^2/\text{s}$, $g = 9.81 \text{ m/s}^2$. Compute the appropriate pipe diameter.

Let's recall the transformed Colebrook-white equation as follows:

$$F(z) = \alpha z^5 + \ln(\beta z^2 + \gamma z^3) \quad (21)$$

The three parameters of the transformed Eq. (21) are as follows.

According to Eq. (16), the following result is obtained:

$$\alpha = \frac{\sqrt{2} Q \ln(10)}{\pi \sqrt{g S_f}} \approx 1.40404996396204 \text{ m}^{5/2} \quad (16)$$

Eq. (17) gives what follows:

$$\beta = \frac{\varepsilon}{3.7} \approx 0.00027027027027 \text{ m} \quad (17)$$

and

$$\gamma = \frac{2.51\nu}{\sqrt{2gS_f}} = 0.00000801381325301697 \text{ m}^{3/2} \quad (18)$$

1. Calculation of the exact pipe diameter using Halley's method

By substituting the previously calculated parameter values into Eq. (21) and applying Halley's method, the exact value of z is obtained as follows:

$$z = 1.39827764350974 \text{ m}^{-1/2} \quad (98)$$

Therefore, Eq. (15) gives the exact sought pipe diameter as follows:

$$D = \frac{1}{z^2} \approx 0.5114617 \text{ m} \quad (15)$$

This is the required pipe diameter for the prescribed discharge and friction slope.

2. Calculation of the diameter using the full rough expansion

If the flow is in the fully rough regime, inequality (69) continues to hold.

$$\frac{\gamma z_R}{\beta} \ll 1 \quad (69)$$

The variable z_R in this regime is given by Eq. (45), recalled as follows:

$$z_R = \left[\frac{2}{5\alpha} W_0 \left(\frac{5\alpha}{2} \beta^{-5/2} \right) \right]^{1/5} \quad (45)$$

Applying an iterative procedure yields the following exact value of the principal branch W_0 :

$$W_0 \left(\frac{5\alpha}{2} \beta^{-5/2} \right) = W_0 (2922986985) = 18.8588875811 \quad (99)$$

Alternatively, the explicit approximation given by Eq. (59) may be used to obtain a quasi-exact value of the principal branch W_0 . Specifically, Eq. (59) yields:

$$W_0(\text{approximate}) = W_0 (2922986985) = 18.85891396 \quad (100)$$

Thus, in the present case, the approximate expression of Eq. (59) reproduces the exact result with a deviation of less than 0.00014%.

Eq. (45) provides the following final result:

$$z_R = \left[\frac{2}{5\alpha} W_0 \left(\frac{5\alpha}{2} \beta^{-5/2} \right) \right]^{1/5} \approx 1.399712046 \text{ m}^{-1/2} \quad (101)$$

Thus, the ratio expressed in the inequality (69) is as follows:

$$\frac{\gamma z_R}{\beta} = 0.0417890 \quad (69a)$$

It follows that the value of the dimensionless ratio under consideration represents only 4.1% of unity and is therefore much smaller than unity. Consequently, inequality (69) is satisfied, indicating that flow resistance is governed by the fully rough turbulent regime under the prescribed hydraulic conditions.

Using Eq. (15), the first value of the rough-pipe diameter is obtained as follows:

$$D_R = \frac{1}{z_R^2} \approx 0.510414026 \text{ m} \quad (15)$$

Thus, according to Eq. (81), the sought corrected full rough pipe diameter is given as follows;

$$D \approx \left[z_R - \frac{\ln \left(1 + \frac{\gamma z_R}{\beta} \right)}{5\alpha z_R^4 + \frac{2}{z_R} + \frac{\gamma}{\beta + \gamma z_R}} \right]^{-2} \approx 0.5114597 \text{ m} \quad (81)$$

Compared with the exact diameter determined by Halley's method, the relative deviation is only 0.00038%.

3. Calculation of the diameter using the hydraulically smooth expansion

In this regime, the inequality (86) must be satisfied. Thus, the following can be written:

$$\frac{\beta}{\gamma z_S} \ll 1 \quad (86)$$

n this regime, the variable z_S is defined by Eq. (56), reproduced below:

$$z_S = \left[\frac{3}{5\alpha} W_0 \left(\frac{5\alpha}{3} \gamma^{-5/3} \right) \right]^{1/5} \quad (56)$$

Using Halley's iterative method, the exact value of W_0 is obtained as follows.

$$W_0 \left(\frac{5\alpha}{3} \gamma^{-5/3} \right) = W_0 (729176419.9) \approx 17.54278359 \quad (102)$$

Once again, the explicit approximation given by Eq. (59) may be safely used to obtain a quasi-exact value of W_0 . In the present case, Eq. (59) yields the following result:

$$W_0(\text{approximate}) = W_0 (729176419.9) \approx 17.54281562 \quad (103)$$

The approximate expression given by Eq. (59) then yields a relative deviation of less than 0.000183%.

Thus, Eq. (56) provides the following final result:

$$z_S = \left[\frac{3}{5\alpha} W_0 \left(\frac{5\alpha}{3} \gamma^{-5/3} \right) \right]^{1/5} \approx 1.49614415 \text{ m}^{-1/2} \quad (104)$$

The ratio expressed by the inequality (86) is as follows:

$$\frac{\beta}{\gamma z_R} = 22.5416457 \gg 1 \quad (105)$$

It follows that the dimensionless ratio under consideration is substantially greater than unity. Consequently, the criterion defined by inequality (86) is unequivocally violated, demonstrating that the hydraulically smooth-flow regime is no longer the dominant resistance mechanism under the prescribed hydraulic conditions. Under these conditions, the influence of wall roughness can no longer be neglected, and the flow departs significantly from the ideal hydraulically smooth limit.

4. Validation using the Rough Model Method (RRM)

The approximate diameter D_0 is determined from the following fundamental RMM relationship.

$$D_0 = \psi D^* \quad (26)$$

where the dimensionless parameter ψ is called “the dimensionless correction factor of linear dimensions.

Next, Eq. (29) is used to compute D^* , yielding the following result:

$$D^* = \left(\frac{Q^2}{2\pi^2 g S_f} \right)^{1/5} = 0.621805775 \text{ m} \quad (29)$$

The parameter ψ is recommended to be evaluated according to the following relationship, which is valid under fully rough turbulent flow conditions.

$$\psi(D^*, \varepsilon) \approx \left\{ \frac{1}{5 \ln(10)} W_0 \left[5 \ln(10) \left(\frac{3.7 D^*}{\varepsilon} \right)^{5/2} \right] \right\}^{-2/5} \quad (27a)$$

The argument is as follows:

$$A = 5 \ln(10) \left(\frac{3.7 D^*}{\varepsilon} \right)^{5/2} = 2922986983$$

Application of Halley's method yields the following exact value of W_0 :

$$W_0(2922986983) = 18.858887577465705$$

It should be emphasized that the explicit form of Eq. (59) provides a highly accurate approximation of the principal branch of the Lambert W function, W_0 , with a maximum relative deviation of only 0.00014%, thereby allowing it to be employed with complete confidence.

Thus, Eq. (27a) gives the following result:

$$\psi = 0.820857647$$

Furthermore, the approximate diameter D_0 is obtained from the fundamental RMM equation (26) as follows:

$$D_0 = \psi D^* = 0.510414026 \text{ m} \quad (26)$$

However, this value of D_0 is only a preliminary estimate, as it is derived from the correction factor ψ corresponding to the fully rough turbulent-flow regime [Eq. (27a)]. Consequently, it must be corrected to account for the residual influence of the smooth turbulent-flow regime, even though this influence is relatively small. The corrected diameter is then obtained by applying Eq. (81), recalled as follows:

$$D \approx \left[z_R - \frac{\ln \left(1 + \frac{\gamma z_R}{\beta} \right)}{5\alpha z_R^4 + \frac{2}{z_R} + \frac{\gamma}{\beta + \gamma z_R}} \right]^{-2} \quad (81)$$

According to Eq. (15), the following can be obtained:

$$z_R = \frac{1}{\sqrt{D_0}} = 1.399712046 m^{-1/2} \quad (15)$$

Eq. (81) provides the following final result:

$$D \approx \left[z_R - \frac{\ln \left(1 + \frac{\gamma z_R}{\beta} \right)}{5\alpha z_R^4 + \frac{2}{z_R} + \frac{\gamma}{\beta + \gamma z_R}} \right]^{-2} = 0.511459753 m \quad (81a)$$

This is the sought value of the considered rough pipe diameter.

Accordingly, the relative deviation between the exact pipe diameter obtained in subsection 1, using Halley's method, and the approximate value predicted by Eq. (81) through the Rough Model Method (RMM) is only 0.00038%. Such an exceedingly small deviation is well within the accuracy requirements of the vast majority of practical hydraulic engineering applications.

In light of this remarkable result, the authors wish to draw attention to the following observations, which are essential for understanding the hydraulic engineering practice.

From an engineering design perspective, it should be emphasized that determining the theoretical pipe diameter to machine-level precision is rarely a practical requirement. The diameter obtained from hydraulic calculations represents an ideal theoretical value, whereas commercially manufactured pipes are available only in a discrete range of standardized nominal sizes. Consequently, the computed diameter serves primarily as a design criterion for selecting the smallest standardized pipe size that is equal to or greater than the theoretical requirement, thereby ensuring compliance with the prescribed hydraulic performance.

Accordingly, the extremely small differences observed between the approximate and the exact analytical solutions are, in almost all practical engineering applications, of no tangible significance. Relative deviations amounting to only a few hundredths, or even a few thousandths, of a percent do not affect the final design decision, since the selected pipe diameter is ultimately governed by standardized manufacturing dimensions rather than by the exact numerical value predicted by the mathematical model. Therefore, once

the analytical approximation leads to the same standardized commercial diameter as the exact solution, any additional numerical refinement becomes practically irrelevant from a design standpoint.

Example 2

Let's consider the following data:

$Q = 0.20 \text{ m}^3/\text{s}$, $S_f = 0.001$, $\varepsilon = 0.000001 \text{ m}$, $\nu = 0.000001 \text{ m}^2/\text{s}$, $g = 9.81 \text{ m/s}^2$. Compute the appropriate pipe diameter.

Let's recall the transformed Colebrook-white equation as follows:

$$F(z) = \alpha z^5 + \ln(\beta z^2 + \gamma z^3) \quad (21)$$

The three parameters of the transformed Eq. (21) are as follows.

According to Eq. (16), the following result is obtained:

$$\alpha = \frac{\sqrt{2} Q \ln(10)}{\pi \sqrt{g S_f}} \approx 2.09303410881684 \text{ m}^{5/2} \quad (16)$$

Eq. (17) gives what follows:

$$\beta = \frac{\varepsilon}{3.7} \approx 0.00000027027027 \text{ m} \quad (17)$$

and

$$\gamma = \frac{2.51 \nu}{\sqrt{2 g S_f}} = 0.00001791943119 \text{ m}^{3/2} \quad (18)$$

1. Calculation of the exact pipe diameter using Halley's method

Substituting the previously calculated parameter values into Eq. (21) and solving it using Halley's method yields the following exact value of z :

$$z = 1.36673055951177 \text{ m}^{-1/2} \quad (106)$$

Therefore, Eq. (15) gives the exact sought pipe diameter as follows:

$$D = \frac{1}{z^2} \approx 0.53534554095704 \text{ m} \quad (15)$$

This is the required pipe diameter for the prescribed discharge and friction slope.

2. Calculation of the diameter using the full rough expansion

In the fully rough regime, inequality (69) remains valid.

$$\frac{\gamma z_R}{\beta} \ll 1 \quad (69)$$

The variable z_R in this regime is given by Eq. (45), recalled as follows:

$$z_R = \left[\frac{2}{5\alpha} W_0 \left(\frac{5\alpha}{2} \beta^{-5/2} \right) \right]^{1/5} \quad (45)$$

Using the Halley's iterative method, the exact value of the principal branch of the Lambert W function, W_0 , is obtained as follows:

$$W_0 \left(\frac{5\alpha}{2} \beta^{-5/2} \right) = W_0 (137790927872806000) = 35.8842163724044 \quad (107)$$

However, the explicit approximation given by Eq. (59) may be used to obtain a quasi-exact value of W_0 . Specifically, Eq. (59) yields the following result:

$$W_0(\text{approximate}) = W_0 (137790927872806000) = 35.88422011 \quad (108)$$

Thus, in the present case, the approximate expression given by Eq. (59) yields a relative deviation of less than 0.0000105%.

Eq. (45) provides the following final result:

$$z_R = \left[\frac{2}{5\alpha} W_0 \left(\frac{5\alpha}{2} \beta^{-5/2} \right) \right]^{1/5} \approx 1.469729565 \text{ m}^{-1/2} \quad (109)$$

Thus, the ratio expressed in the inequality (69) is as follows:

$$\frac{\gamma z_R}{\beta} = 97.445855 \gg 1 \quad (69)$$

It follows that the dimensionless ratio under consideration is substantially greater compared with unity. As a consequence, the condition specified by inequality (69) is not fulfilled, implying that the fully rough turbulent-flow regime is not the prevailing resistance mechanism under the prescribed hydraulic conditions. Consequently, further analysis based on this flow regime is neither justified nor necessary, and the corresponding calculations need not be pursued.

3. Calculation of the diameter using the hydraulically smooth expansion

In this regime, the inequality (86) must be satisfied.

$$\frac{\beta}{\gamma z} \ll 1 \quad (86)$$

The variable z_S in this regime is given by Eq. (56), recalled as follows:

$$z_S = \left[\frac{3}{5\alpha} W_0 \left(\frac{5\alpha}{3} \gamma^{-5/3} \right) \right]^{1/5} \quad (56)$$

Using an iterative procedure, bases on Halley's method, the exact value of the principal branch of the Lambert W function, W_0 , is obtained as follows:

$$W_0 \left(\frac{5\alpha}{3} \gamma^{-5/3} \right) = W_0 (284283304.808348) \approx 16.65289761861 \quad (110)$$

Once again, the explicit approximation given by Eq. (59) may be safely used to obtain a quasi-exact value of W_0 . In the present case, Eq. (59) yields the following result:

$$W_0 (\text{approximate}) = W_0 (284283304.808348) \approx 16.6529343 \quad (111)$$

The approximate Eq. (59) then produces a deviation less than 0.00023%.

Eq. (56) provides the value of z_S as follows:

$$z_S = \left[\frac{3}{5\alpha} W_0 \left(\frac{5\alpha}{3} \gamma^{-5/3} \right) \right]^{1/5} \approx 1.367013965346 \text{ m}^{-1/2} \quad (112)$$

Thus, the ratio expressed by the inequality (86) is as follows:

$$\frac{\beta}{\gamma z} = 0.01103319.5416457 \ll 1 \quad (113)$$

Since this ratio is only about 1.1% of unity, the roughness contribution is very small compared with the viscous contribution. The hydraulically smooth resistance mechanism may therefore be regarded as dominant for the selected data.

Using Eq. (97) the sought diameter of the hydraulically smooth pipe is given as follows:

$$D \approx \left[z_S - \frac{\ln\left(1 + \frac{\beta}{\gamma z_S}\right)}{5\alpha z_S^4 + \frac{2}{z_S} + \frac{\gamma}{\beta + \gamma z_S}} \right]^{-2} \approx 0.53534546 \text{ m} \quad (97)$$

Consequently, the pipe diameter predicted by Eq. (97) is virtually indistinguishable from the exact solution of the governing Eq. (21), the relative deviation being as low as 0.000016%. This exceptionally small discrepancy demonstrates the remarkable accuracy of the proposed analytical formulation.

4. Validation using the Rough Model Method (RRM)

Let's recall the equations derived from the Rough Model Method (RMM). The approximate diameter D_0 to be adopted is given by the following RMM fundamental relationship:

$$D_0 = \psi D^* \quad (26)$$

where the dimensionless parameter ψ is called “the dimensionless correction factor of linear dimensions”.

The related diameter D^* is explicitly given as follows:

$$D^* = \left(\frac{Q^2}{2\pi^2 g S_f} \right)^{1/5} \quad (29)$$

Thus, for the given data, Eq. (29) provides the following result:

$$D^* = 0.72947808 \text{ m}$$

Therefore, according to Eq. (28), the Reynolds number R^* is as follows:

$$R^* = 349082.33$$

For the fully hydraulically smooth turbulent-flow regime, the authors recommend evaluating the parameter ψ using the following explicit analytical expression:

$$\psi(R^*) \approx \left\{ \frac{3}{10\ln(10)} W_0 \left[\frac{10\ln(10)}{3} \left(\frac{R^*}{10.04} \right)^{5/3} \right] \right\}^{-2/5} \quad (114)$$

W_0 can also be computed explicitly using the approximate expression given by Eq. (59), yielding a relative deviation of only 0.00022%.

The argument of W_0 in Eq. (114) is given by the following:

$$A = \frac{10 \ln(10)}{3} \left(\frac{R^*}{10.04} \right)^{5/3} = 284283305 \quad (115)$$

Thus, application of Halley's iterative method yields the following result:

$$W_0^{\text{Exact}}(A) = W_0(284283305) = 16.65289762 \quad (116)$$

Furthermore, according to Eq. (114), the dimensionless factor of linear dimensions ψ is obtained as follows:

$$\psi = 0.73357048$$

Consequently, the approximate sought diameter is given by the RMM fundamental Eq. (26) as follows:

$$D_0 = \psi D^* = 0.53512359 \text{ m} \quad (26)$$

The Rough Model Method (RMM) predicts the hydraulically smooth pipe diameter with a relative deviation of only 0.0415%.

It should be emphasized that the previous diameter D_0 obtained at this stage does not yet represent the complete solution. Indeed, the parameter ψ , expressed by Eq. (114), has been evaluated under the assumption of a fully hydraulically smooth turbulent-flow regime ($\varepsilon = 0$), thereby neglecting the residual influence of wall roughness. Although the residual influence of wall roughness is expected to be very small under the present hydraulic conditions, it nevertheless remains finite. Accordingly, it must be systematically reintroduced through the analytical correction provided by Eq. (97), thereby restoring the complete transformed Colebrook-White solution while preserving the physical consistency of the governing model.

Thus, using the value of D_0 previously calculated, z_S is given by Eq. (15) as follows:

$$z_S = \frac{1}{\sqrt{D_0}} = 1.367013965 \text{ m}^{-1/2} \quad (15)$$

Finally, the sought hydraulically smooth pipe diameter is obtained from Eq. (97) as follows:

$$D \approx \left[z_S - \frac{\ln \left(1 + \frac{\beta}{\gamma z_S} \right)}{5\alpha z_S^4 + \frac{2}{z_S} + \frac{\gamma}{\beta + \gamma z_S}} \right]^{-2} \approx 0.535345455 \text{ m} \quad (97)$$

Consequently, the diameter predicted by Eq. (97) is virtually indistinguishable from the exact solution, the corresponding relative deviation being less than 0.000016%. Such an exceptionally small discrepancy confirms the remarkable predictive capability of the proposed RMM analytical formulation and renders any further numerical refinement practically unnecessary for engineering design purposes.

CONCLUSION

This study has presented a new analytical framework for the direct determination of the diameter of circular pressurized pipes governed by the exact Darcy-Weisbach and Colebrook-White equations. Unlike conventional approaches, which rely on repeated coupled iterations involving the pipe diameter, the Reynolds number, and the Darcy friction factor, the proposed methodology transforms the original hydraulic problem into a single nonlinear scalar equation containing only one unknown variable. This analytical reformulation preserves the complete physical content of the governing equations while considerably simplifying their mathematical structure.

A rigorous mathematical analysis has demonstrated that the transformed equation is continuous and strictly increasing over the physically admissible domain. Consequently, the existence and uniqueness of the positive solution are established without ambiguity, providing a solid theoretical foundation for the proposed computational procedure.

The combination of the Rough Model Method with Halley's third-order iterative algorithm has proved to be particularly effective. The analytical approximation supplied by the RMM already reproduces the exact diameter with extremely small relative deviations, making it suitable for most practical hydraulic engineering applications without any iterative computation. When higher numerical accuracy is required, a single Halley correction is generally sufficient to recover the exact solution with near machine precision. In this formulation, Halley's method no longer acts as a conventional iterative solver but rather as a final analytical refinement of an already highly accurate initial estimate.

The asymptotic analysis has further demonstrated that the transformed equation naturally admits exact closed-form solutions in the fully rough and hydraulically smooth turbulent limits. These solutions, expressed through the principal branch of the Lambert W function, recover the expected physical behavior of both limiting regimes. In the fully rough limit, the final diameter becomes completely independent of the Reynolds number and the fluid viscosity, whereas, in the hydraulically smooth limit, the absolute roughness disappears entirely from the governing expression. These results constitute direct analytical confirmations of the classical theory of turbulent wall resistance.

The introduction of a high-accuracy analytical approximation of the principal branch W_0 further enhances the practical applicability of the proposed asymptotic solutions by eliminating the need for iterative evaluation of the Lambert function itself. The resulting explicit expressions reproduce the exact Lambert-based solutions with deviations that are negligible for engineering design purposes.

Finally, the study has shown that, although the complete transformed Colebrook-White diameter equation does not admit a closed-form solution in terms of currently known elementary or Lambert-type functions, it nevertheless possesses a well-defined asymptotic analytical structure. The proposed series-based treatment establishes a rigorous bridge between the general turbulent-flow solution and its exact asymptotic limits by introducing the neglected resistance mechanism through successive logarithmic corrections. This interpretation provides a deeper understanding of the mathematical nature of the transformed equation and opens new perspectives for the analytical investigation of nonlinear logarithmic resistance laws.

As concluding sentences, the proposed methodology combines mathematical rigor, physical consistency, computational efficiency, and practical simplicity within a unified analytical framework. Beyond its immediate application to hydraulic pipe design, the analytical concepts developed in this work may provide a foundation for future investigations devoted to the solution of broader classes of nonlinear logarithmic and polynomial-exponential equations encountered in fluid mechanics and related engineering disciplines.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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