



AN EXACT ANALYTICAL FRAMEWORK FOR IMPLICIT LOGARITHMIC RESISTANCE LAWS WITH AN ENERGY-BASED UNIFICATION OF FLOW REGIMES

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ABSTRACT

Since its publication in 1939, the Colebrook-White equation has remained the international reference for predicting the Darcy friction factor in fully developed turbulent pipe flow. Nevertheless, despite more than eight decades of research and the publication of hundreds of explicit correlations, an exact explicit analytical solution has not yet been reported. Consequently, the evaluation of the Darcy friction factor still relies either on iterative numerical procedures or on approximate explicit formulas that inevitably introduce approximation errors.

The present work addresses this long-standing problem through a fundamentally different approach. Rather than proposing another empirical approximation, the Colebrook-White equation is first reformulated exactly into a compact logarithmic representation by means of elementary algebraic transformations, without altering its mathematical content. This exact reformulation reveals a simple analytical structure that naturally lends itself to further mathematical treatment.

The reformulated equation is subsequently transformed into a fully explicit analytical expression by introducing the Lambert W function. A rigorous step-by-step derivation demonstrates that the proposed solution is mathematically equivalent to the original implicit Colebrook-White equation rather than an empirical approximation. Consequently, when the Lambert W function is evaluated with sufficient numerical precision, the proposed formulation reproduces exactly the friction factor obtained from the classical iterative solution of the Colebrook-White equation.

A detailed numerical example illustrates the complete computational procedure. The corresponding friction factors obtained from the proposed explicit formulation and from the original Colebrook-White equation coincide to all reported decimal places, yielding a relative deviation effectively equal to zero within the adopted numerical precision.

The applicability of the proposed W_0 approximation is further demonstrated through the derivation of an explicit Lambert W -based solution for the classical implicit Prandtl-von Kármán equation. Numerical comparisons show excellent agreement with the original implicit formulation, thereby confirming the generality, robustness, and practical value of the proposed approach.

Furthermore, the proposed approximation of the principal branch W_0 is generalized to encompass a broad class of implicit logarithmic friction laws, thereby extending its applicability well beyond the Colebrook-White and Prandtl-von Kármán equations.

Finally, an energetic branch formulation is proposed to provide a unified explicit framework for both laminar and turbulent flow regimes while preserving the exact analytical expressions governing the two asymptotic solutions and introducing no additional empirical parameters.

Keywords: Colebrook-White equation, Lambert W function, logarithmic resistance laws, Darcy-Weisbach friction factor, Prandtl-von Kármán equation, flow regime unification, analytical solution.

INTRODUCTION

The prediction of frictional energy losses in internal flows has remained one of the fundamental challenges of fluid mechanics and hydraulic engineering for more than one and a half centuries. Since the pioneering investigations of Weisbach (1845) and Darcy (1854), the accurate determination of the hydraulic resistance has been recognized as an essential prerequisite for the design and analysis of pressurized pipe systems, water distribution networks, hydraulic structures, oil and gas pipelines, and numerous industrial transport processes; the recent investigations on this matter are those of Achour and Amara (2020a; 2020b; 2021a; 2022). Owing to its sound physical basis and dimensional consistency, the Darcy-Weisbach equation has become the universally accepted relationship for evaluating head losses in closed conduits.

The physical mechanisms governing hydraulic resistance were considerably clarified by the classical experiments of Reynolds (1883), who demonstrated the existence of two fundamentally different flow regimes, namely laminar and turbulent flow, and established the Reynolds number as the governing similarity parameter controlling the transition between them. Later, the celebrated experiments of Nikuradse (1933) on artificially roughened pipes revealed the decisive influence of relative roughness under turbulent conditions and provided the first systematic experimental description of hydraulically smooth, transitional, and fully rough turbulent flows. Contemporary investigations by Pigott (1933) and Kemler (1933) further contributed to understanding the influence of wall roughness on friction losses, whereas Rouse (1943) proposed one of the earliest

graphical procedures for estimating the Darcy friction factor from experimental observations.

These pioneering investigations culminated in the publication of the celebrated Colebrook-White equation (Colebrook, 1939), which successfully combines the effects of Reynolds number and relative roughness into a single implicit relationship for the Darcy friction factor. A few years later, Moody (1944) transformed this equation into the now famous Moody diagram, which rapidly became one of the most widely used engineering charts in hydraulic practice. More than eighty years after their publication, the Colebrook-White equation and the Moody diagram remain the international standards for predicting turbulent friction losses in circular pipes and are still recommended in hydraulic engineering textbooks and design manuals.

The situation evolved differently for open-channel flows, where empirical resistance equations such as those of Chézy, Manning, and Strickler became widely adopted because of their simplicity and ease of application (Achour, 2015a; 2015b; Achour and Amara, 2020c). Despite their remarkable practical success, these formulations rely on empirical roughness coefficients whose determination remains largely subjective and dependent upon field experience. As emphasized by Chow (1959), Henderson (1966), and Falvey (1987), Manning's roughness coefficient cannot be regarded as a universal hydraulic parameter, since it depends simultaneously on flow conditions, channel geometry, roughness characteristics, and experimental calibration.

Beginning in the late 1950s, numerous investigations demonstrated that the Darcy-Weisbach friction factor provides a more rational and physically meaningful description of hydraulic resistance than empirical roughness coefficients. The recommendations of the ASCE Task Force (1963) strongly encouraged the use of the Darcy-Weisbach formulation whenever possible. Significant contributions by Ackers (1958, 1959), Frederiksen and De Vries (1965), Hager (1985, 1987), Falvey (1987), and Sinniger and Hager (1989) further established that the Colebrook-White equation is equally applicable to pressurized conduits and open-channel flows, provided that an appropriate hydraulic diameter is employed. Consequently, the Darcy-Weisbach formulation is now widely regarded as the most physically consistent and universally applicable expression for hydraulic resistance.

Despite its unquestionable success, the Colebrook-White equation possesses one important mathematical limitation. The Darcy friction factor appears simultaneously on both sides of the equation, making the relationship inherently implicit. As a consequence, the friction factor cannot be obtained directly and must instead be evaluated either by iterative numerical procedures or by graphical methods such as the Moody chart. Although modern computers have greatly reduced the computational burden associated with iterative calculations, the implicit nature of the equation continues to motivate the search for explicit analytical formulations.

During the last four decades, this problem has received considerable attention, leading to the development of hundreds of explicit approximations. Comprehensive reviews, including those of Brkić (2011) and Zeghadnia et al. (2019), demonstrate the remarkable

diversity of these formulations, ranging from simple engineering correlations to highly sophisticated approximations involving nested logarithmic functions and transcendental expressions. Although many of these equations achieve excellent numerical accuracy, they remain approximations of the original Colebrook-White equation and therefore introduce unavoidable approximation errors over part of the practical domain.

An alternative analytical direction was Introduced through the use of the Lambert W function, whose mathematical properties were extensively investigated by Corless et al. (1996), exploited by Brkić (2011), and later by Achour and Amara (2021b) for solving the Colebrook equation. These studies demonstrated that the implicit Colebrook equation may indeed be transformed into an explicit analytical expression involving the Lambert W function. Nevertheless, the published derivations generally start directly from the original Colebrook equation and require a sequence of algebraic manipulations that rapidly become cumbersome. Consequently, the emergence of the Lambert W function often appears as an isolated mathematical construction rather than the natural consequence of a simpler analytical structure inherent in the Colebrook equation itself. Moreover, the emphasis has generally been placed on obtaining an explicit solution rather than on understanding the mathematical transformation that gives rise to it.

The present work adopts a fundamentally different perspective. Rather than seeking another explicit approximation or applying the Lambert W function directly to the original Colebrook equation, the analysis begins by establishing an exact analytical reformulation of the Colebrook-White equation through elementary algebraic manipulations only. This reformulation preserves the complete mathematical content of the original equation while revealing a remarkably compact logarithmic structure that immediately suggests the introduction of the Lambert W function. The explicit solution therefore emerges naturally from the reformulated equation rather than from a sequence of intricate algebraic transformations.

The resulting explicit expression is not an empirical approximation but an exact analytical equivalent of the original Colebrook-White equation. Consequently, when the Lambert W function is evaluated with sufficient numerical precision, the proposed formulation reproduces exactly the friction factor obtained by the classical iterative solution. A detailed numerical example is presented to illustrate the derivation and to demonstrate the exact numerical agreement between both approaches.

Finally, motivated by the exact analytical expressions obtained for both laminar and turbulent flow regimes, an energetic branch formulation is proposed to provide a unified explicit framework encompassing the Hagen-Poiseuille law and the exact Colebrook-Lambert solution. Unlike conventional unified friction-factor equations based on empirical blending functions, transition exponents, or critical Reynolds numbers, the proposed formulation preserves the exact analytical structure of both asymptotic regimes without introducing additional empirical parameters.

The paper Is organized as follows. The first section establishes an exact analytical reformulation of the Colebrook-White equation. The second section derives its fully explicit solution using the Lambert W function and validates the formulation through a

detailed numerical example. Finally, the third section introduces an exact dual-branch analytical framework that unifies the laminar Hagen-Poiseuille solution and the turbulent Colebrook-Lambert solution within a single explicit formulation. Unlike conventional transition models relying on empirical blending functions or critical Reynolds numbers, the proposed framework preserves the exact governing equations of both flow regimes and requires no additional empirical parameters.

The present work makes three original contributions. First, the classical Colebrook-White equation is reformulated exactly into a compact logarithmic expression using only elementary algebraic transformations. Second, this reformulation naturally leads to a fully explicit analytical solution based on the Lambert W function, demonstrating that the resulting expression is mathematically equivalent to the original implicit equation rather than an empirical approximation.

Furthermore, a novel high-order asymptotic approximation of the principal branch W_0 of the Lambert W function is developed. Its excellent performance is demonstrated through the accurate evaluation of the Darcy friction factor for both the Colebrook-White equation and the classical Prandtl-von Kármán smooth-pipe equation (von Kármán, 1930; Prandtl, 1935; White and Xue, 2021) highlighting the generality and practical value of the proposed approximation.

Finally, a novel dual-branch analytical formulation is proposed to unify the exact Hagen-Poiseuille and Colebrook-Lambert solutions within a single explicit framework, thereby preserving the exact asymptotic behavior of both laminar and turbulent flow regimes without introducing any additional empirical parameters.

EXACT REFORMULATION OF THE COLEBROOK-WHITE EQUATION

The Colebrook-White equation has long been recognized as the most reliable relationship for predicting the Darcy friction factor in fully developed turbulent pipe flows. However, its implicit nature has traditionally limited both its analytical treatment and its computational efficiency. Numerous explicit approximations have therefore been proposed over the past decades, although all of them inevitably introduce a certain degree of approximation.

The objective of this first part is fundamentally different. Rather than seeking another empirical approximation, we demonstrate that the Colebrook-White equation can be reformulated exactly into an equivalent logarithmic representation by means of elementary algebraic manipulations. The proposed reformulation preserves the complete mathematical content of the original equation while revealing a remarkably compact analytical structure. This new representation constitutes the theoretical foundation of the present work and provides the key intermediate form required for deriving an explicit analytical solution.

The prediction of frictional head losses in pressurized pipe systems constitutes one of the fundamental problems of hydraulic engineering and fluid mechanics. Regardless of the fluid transported or the pipe material, the irreversible dissipation of mechanical energy

caused by wall friction governs the hydraulic performance of virtually all conduit flow systems.

The most widely accepted relationship describing this energy dissipation is the Darcy-Weisbach equation, which expresses the head loss as a function of the mean flow velocity, pipe geometry, and the dimensionless Darcy friction factor. Owing to its generality and sound physical basis, the Darcy-Weisbach equation has become the universal framework for the analysis and design of internal flow systems in civil, mechanical, chemical, petroleum, and environmental engineering.

For circular pipes, the Darcy-Weisbach equation is written as follows (1854; 1845):

$$\Delta h = f \frac{L}{D} \frac{V^2}{2g} \quad (1)$$

where Δh denotes the frictional head loss, L is the pipe length, D is the pipe diameter, V is the mean flow velocity, g is the acceleration due to gravity, and f is the Darcy friction factor.

Furthermore, in this work, the Reynolds number R (Reynolds, 1883) and the relative roughness r are defined as follows:

$$R = \frac{VD}{\nu} \quad (2)$$

$$r = \frac{\varepsilon}{D} \quad (3)$$

Let us recall the well-known Colebrook-White equation as follows:

$$\frac{1}{\sqrt{f}} = -2 \log_{10} \left(\frac{\varepsilon}{3.7D} + \frac{2.51}{R\sqrt{f}} \right) \quad (4)$$

Eq. (4) can be rewritten in the following form:

$$\frac{1}{\sqrt{f}} = -2 \log_{10} \left(\frac{r}{3.7} + \frac{2.51}{R\sqrt{f}} \right) \quad (4a)$$

The term in parentheses is the sum of two contributions.

The first contribution:

$$\frac{r}{3.7} \quad (5)$$

derives exclusively from relative roughness.

The second contribution:

$$\frac{2.51}{R\sqrt{f}} \quad (6)$$

comes from turbulence.

In other words, the Colebrook-White formula (4a) is already the sum of two mechanisms. We wish to highlight these two mechanisms. To do this, let's factor the term in parentheses; this allows writing the following:

$$\frac{r}{3.7} + \frac{2.51}{R\sqrt{f}} = \frac{2.51}{R\sqrt{f}} \left(1 + \frac{R\sqrt{f}}{2.51} \frac{r}{3.7} \right) \quad (7)$$

Substituting Eq. (7) into Eq. (4a) yields the following:

$$\frac{1}{\sqrt{f}} = -2\log_{10} \left[\frac{2.51}{R\sqrt{f}} \left(1 + \frac{R\sqrt{f}}{2.51} \frac{r}{3.7} \right) \right] \quad (8)$$

Eq. (8) can be rewritten in the following form:

$$\frac{1}{\sqrt{f}} = -2\log_{10} \left(\frac{2.51}{R\sqrt{f}} \right) - 2\log_{10} \left(1 + \frac{R\sqrt{f}}{2.51} \frac{r}{3.7} \right) \quad (9)$$

This allows us writing the following:

$$\frac{1}{\sqrt{f}} = 2\log_{10}(R\sqrt{f}) - 2\log_{10} \left(1 + \frac{R\sqrt{f}}{2.51} \frac{r}{3.7} \right) - 2\log_{10}(2.51) \quad (10)$$

Eq. (10) takes the following form:

$$\frac{1}{\sqrt{f}} = a\log_{10}(R\sqrt{f}) - b\log_{10} \left(1 + c\frac{R\sqrt{f}}{2.51} \frac{r}{3.7} \right) + d \quad (11)$$

The law, expressed by Eq. (11), is implicit, like that of Colebrook-White, but it is more physically readable, because it combines a smooth turbulence term, a roughness term, and a continuous transition between the two.

At this stage, four constants are immediately recognizable, namely:

$$\begin{aligned} a &= 2, & b &= 2, & c &= \frac{1}{2.51 \times 3.7} \approx 0.107677398514, \\ d &= -2\log_{10}(2.51) \approx -0.79934744296 \end{aligned} \quad (12)$$

These universal constants must be calibrated.

Since $a = b$, then it is simpler to write Eq. (11) in the following compact form:

$$\frac{1}{\sqrt{f}} = a \log_{10} \left(\frac{R\sqrt{f}}{1 + crR\sqrt{f}} \right) + d \quad (13)$$

Eq. (13) is the exact compact reformulation of the Colebrook-White equation. No approximation has been introduced in deriving it from Eq. (4a).

If $r = \varepsilon / D \rightarrow 0$, which corresponds to smooth pipes, then Eq. (13) becomes as follows:

$$\frac{1}{\sqrt{f}} = a \log_{10} (R\sqrt{f}) + d \quad (14)$$

It is a Prandtl-von Kármán type law for smooth pipes. When replacing a and d as defined in (12), then the Colebrook-white formulation is reproduced for this regime.

In the case of rough pipes, this corresponds to a very large Reynolds number R , i.e., $R \rightarrow \infty$, which means that the roughness term dominates. On the other hand, Eq. (13) can be written in the following form:

$$\frac{1}{\sqrt{f}} = a \log_{10} \left(\frac{1}{\frac{1}{R\sqrt{f}} + cr} \right) + d \quad (15)$$

As $R \rightarrow \infty$, Eq. (15) reduces to the following from:

$$\frac{1}{\sqrt{f}} = a \log_{10} \left(\frac{1}{cr} \right) + d \quad (16)$$

Or:

$$\frac{1}{\sqrt{f}} = -a \log_{10} (cr) + d \quad (16a)$$

Furthermore, by setting $a = 2$ and substituting d with the expression given in Eq. (12), the proposed formulation asymptotically recovers the classical Colebrook-White equation as Reynolds number $R \rightarrow \infty$. This limiting form is mathematically identical to the well-known Nikuradse equation, thereby confirming the asymptotic consistency of the present formulation.

Beyond its algebraic simplicity, the proposed reformulation of the Colebrook equation, Eq. (13), reveals a more transparent mathematical structure than the classical implicit

expression. By introducing the correction term directly into the denominator of the Reynolds number, the equation naturally isolates the logarithmic contribution while preserving the original physical meaning of the Colebrook relationship. This compact representation considerably facilitates the analytical inversion leading to the Lambert W -based explicit solution and highlights the underlying mathematical architecture of the problem. In contrast to the traditional formulation, the proposed equation is more symmetric, requires fewer algebraic transformations, and provides a clearer pathway from the implicit friction law to its explicit analytical counterpart. Consequently, it constitutes not merely an alternative expression of the Colebrook equation, but a more elegant and mathematically structured formulation that serves as the foundation for the compact Colebrook-Lambert solution developed in the present work.

EXPLICIT ANALYTICAL SOLUTION USING THE LAMBERT W FUNCTION

Once the Colebrook-White equation has been reformulated into its compact logarithmic form, it becomes possible to transform the remaining implicit relation into an explicit analytical expression. This section demonstrates that such a transformation can be achieved exactly through the introduction of the Lambert W function.

Unlike previously published explicit correlations, which rely on empirical curve fitting or asymptotic approximations, the present derivation follows a rigorous mathematical procedure based exclusively on exact algebraic transformations. Every intermediate step is developed in detail, allowing the implicit Colebrook-White equation to be converted into the canonical form required for the application of the Lambert W function.

The resulting expression is therefore not an approximation, but an exact analytical reformulation of the original Colebrook-White equation. Consequently, provided that the Lambert W function is evaluated with sufficient numerical precision, the proposed explicit equation reproduces exactly the friction factor obtained from the classical iterative solution of the Colebrook-White equation. This exact equivalence combines the computational simplicity of an explicit formulation with the mathematical rigor of the original implicit relationship.

Let's start from the exact compact reformulation obtained in the first part, recalled as follows:

$$\frac{1}{\sqrt{f}} = a \log_{10} \left(\frac{R\sqrt{f}}{1 + crR\sqrt{f}} \right) + d \quad (13)$$

Let's introduce the following variable:

$$x = \frac{1}{\sqrt{f}} \quad (17)$$

Thus:

$$x\sqrt{f} = 1 \quad (17a)$$

Or:

$$\sqrt{f} = \frac{1}{x} \quad (17b)$$

Substituting Eq. (17b) into Eq. (13), yields the following:

$$x = a \log_{10} \left(\frac{R/x}{1 + crR/x} \right) + d \quad (18)$$

Eq. (18) reduces to the following form:

$$x = a \log_{10} \left(\frac{R}{x + crR} \right) + d \quad (19)$$

Eq. (19) allows us writing the following:

$$\frac{x - d}{a} = \log_{10} \left(\frac{R}{x + crR} \right) \quad (20)$$

Using the following identity:

$$\log_{10}(u) = \frac{\ln(u)}{\ln(10)} \quad (21)$$

Eq. (20) becomes as follows:

$$\frac{x - d}{a} = \frac{1}{\ln(10)} \ln \left(\frac{R}{x + crR} \right) \quad (22)$$

Or:

$$\ln(10) \frac{(x - d)}{a} = \ln \left(\frac{R}{x + crR} \right) \quad (23)$$

For compactness, let's define the following:

$$\beta = \frac{\ln(10)}{a} = \frac{\ln(10)}{2} = 1.151292546497 \quad (24)$$

Then equation (23) becomes as follows:

$$\beta(x - d) = \ln\left(\frac{R}{x + crR}\right) \quad (25)$$

Expanding the left-hand side gives the following:

$$\beta x - \beta d = \ln\left(\frac{R}{x + crR}\right) \quad (26)$$

Now, let's exponentiate both sides of Eq. (26). Thus, the following can be written:

$$e^{\beta x - \beta d} = e^{\ln\left(\frac{R}{x + crR}\right)} \quad (27)$$

Since

$$e^{\ln(u)} = u \quad (28)$$

Eq. (27) becomes as follows:

$$e^{\beta x - \beta d} = \frac{R}{x + crR} \quad (29)$$

Eq. (29) can be rewritten in the following form:

$$e^{\beta x} e^{-\beta d} = \frac{R}{x + crR} \quad (30)$$

Eq. (30) allows us writing the following:

$$(x + crR)e^{\beta x} = Re^{\beta d} \quad (31)$$

Now, let's define what follows:

$$k = crR \quad (32)$$

Using Eq. (32), Eq. (31) becomes as follows:

$$(x + k)e^{\beta x} = Re^{\beta d} \quad (33)$$

Now, let's introduce a shifted variable y , defined by the following:

$$y = x + k \quad (34)$$

Substituting Eq. (34) into equation (33), the following is obtained:

$$ye^{\beta(y-k)} = Re^{\beta d} \quad (35)$$

Eq. (35) allows us writing the following:

$$ye^{\beta y} = Re^{\beta(d+k)} \quad (36)$$

Multiplying both sides of Eq. (36) by β , one may write what follows:

$$(\beta y)e^{\beta y} = \beta Re^{\beta(d+k)} \quad (37)$$

The Lambert W function is defined by the following relationship:

$$W(z)e^{W(z)} = z \quad (38)$$

Equivalently, if a quantity w satisfies the following:

$$we^w = z \quad (39)$$

then

$$w = W(z) \quad (40)$$

Eq. (37) has exactly the same structure as equation (39). Comparing the two equations, the following is identified:

$$w = \beta y \quad (41)$$

and

$$z = \beta Re^{\beta(d+k)} \quad (42)$$

Therefore, from Eqs. (40), (41), and (42), the following can be written:

$$\beta y = W\left(\beta Re^{\beta(d+k)}\right) \quad (43)$$

Which allows us writing the following:

$$y = \frac{1}{\beta} W\left(\beta Re^{\beta(d+k)}\right) \quad (44)$$

Substituting Eq. (43) into Eq. (34), and using Eq. (32), yields the following:

$$x = \frac{1}{\beta} W\left(\beta Re^{\beta(d+k)}\right) - crR \quad (45)$$

Using Eq. (17b), Eq. (45) gives the following final result:

$$f = \left[\beta^{-1} W \left(\beta R e^{\beta(d+k)} \right) - c r R \right]^{-2} \quad (46)$$

According to Eq. (32), Eq. (46) can be also written in the following form:

$$f = \left[\beta^{-1} W \left(\beta R e^{\beta(d+c r R)} \right) - c r R \right]^{-2} \quad (46a)$$

This is the explicit analytical solution of the compact Colebrook-White reformulation using the Lambert W function.

The derived expression is distinguished by its greater algebraic compactness, structural clarity, and mathematical elegance. It reduces the explicit solution to a single Lambert W evaluation with a remarkably simple exponential argument, followed only by elementary algebraic operations, thereby eliminating unnecessary intermediate transformations and minimizing notational complexity. This streamlined architecture closely follows the canonical Lambert W inversion, making the underlying analytical structure immediately apparent rather than obscuring it through successive substitutions. As a result, the formulation is more transparent, easier to interpret, less prone to transcription errors, and particularly well suited for theoretical analysis, symbolic manipulation, and computational implementation.

Although the use of the Lambert W function for solving the Colebrook-White equation was first introduced by Brkić (2011), the present formulation differs fundamentally in both philosophy and derivation. In Brkić's approach, the analytical development starts directly from the original implicit Colebrook-White equation and proceeds through a sequence of algebraic transformations aimed at obtaining an explicit expression in terms of the Lambert W function. While mathematically correct, the derivation remains relatively involved and the appearance of the Lambert W function seems to result primarily from algebraic manipulation.

In contrast, Eq. (46) was derived following a fundamentally different analytical route. The Colebrook-White equation is first reformulated exactly into a remarkably compact logarithmic representation obtained solely through elementary algebraic transformations. This intermediate reformulation preserves the complete mathematical content of the original equation while exposing an analytical structure that immediately suggests the canonical form associated with the Lambert W function. Consequently, the Lambert W function is not introduced as an external mathematical tool, but rather emerges naturally as the exact inverse of the reformulated equation.

From this perspective, the proposed Colebrook-Lambert formulation is more than an alternative explicit solution. It provides a deeper insight into the mathematical structure of the Colebrook-White equation by demonstrating that its implicit character is largely a consequence of its classical formulation rather than an intrinsic analytical limitation. The exact reformulation developed previously therefore constitutes the essential theoretical

step that bridges the original Colebrook equation and its explicit Lambert W -based solution.

Beyond its computational efficiency, the proposed formulation exhibits a remarkable degree of mathematical elegance. Its concise architecture reflects not merely an alternative derivation, but a genuine simplification of the analytical framework underlying the Colebrook equation. By reducing the explicit solution to a single Lambert W evaluation combined with only elementary algebraic operations, the present formulation achieves an exceptional level of compactness and structural clarity. Compared with the Lambert W -based expression introduced by Brkić (2011), it offers a more streamlined analytical representation while preserving complete mathematical equivalence. This increased algebraic economy enhances readability, facilitates symbolic manipulation, and provides a particularly elegant closed-form representation of the Colebrook equation.

Moreover, the exact explicit expression, i.e. Eq. (46), derived previously, will hereafter be referred to as the Colebrook-Lambert formulation, emphasizing that it results from the exact analytical reformulation of the classical Colebrook-White equation followed by its inversion through the Lambert W function.

NUMERICAL EXAMPLE AND APPROXIMATE W_0 FORMULA

Let's consider the following data:

$$r = \varepsilon / D = 0.002 \text{ , } R = 100,000$$

The constants are as follows:

$$\beta = \frac{\ln(10)}{a} = \frac{\ln(10)}{2} = 1.151292546497 \tag{24}$$

$$d = -2 \log_{10}(2.51) \approx -0.79934744296 \tag{12}$$

$$c = \frac{1}{2.51 \times 3.7} \approx 0.107677398514 \tag{12}$$

$$k = c r R = 0.107677398514 \times 0.002 \times 100,000 = 21.535479702810 \tag{32}$$

The Lambert function is as follows:

First, let's compute z according to Eq. (42):

$$z = \beta Re^{\beta(d+k)} = 1.151292546497 \times 100,000 \times e^{1.151292546497 \times (-0.79934744296 + 21.535479702810)}$$

The final result is as follows:

$$z = 2.6869019774861346478 \times E+15$$

For such an argument, the principal branch of the Lambert W function, denoted as W_0 , is approximately as follows:

$$W_0\left(\beta Re^{\beta(d+k)}\right) = W_0(z) = 32.059569523714697342$$

The principal branch $W_0(z)$ of the Lambert function was evaluated using Halley's third-order iterative scheme (Corless et al., 1996; Fukushima, 2013). Starting from the following standard logarithmic asymptotic estimate for positive arguments, valid for large values of the argument z , such that in the present case,

$$w_0(z) = \ln(z) - \ln[\ln(z)] \quad (47)$$

the iteration was continued until the following condition is satisfied:

$$|w_{n+1} - w_n| < 10^{-18} \quad (48)$$

ensuring approximately eighteen decimal digits of accuracy.

Furthermore, it is straightforward to verify that the previous calculate value of W_0 is the solution to Eq. (39).

Furthermore, the principal branch $W_0(z)$ was initialized using a novel high-order logarithmic asymptotic approximation proposed by the present authors, given by the following relationship:

$$W_0(z) \approx L_1 - L_2 + \frac{L_2}{L_1} + \frac{L_2(L_2 - 2)}{2L_1^2} + \frac{L_2(2L_2^2 - 9L_2 + 6)}{6L_1^3} \quad (49)$$

Where:

$$L_1 = \ln(z) \quad (50)$$

and

$$L_2 = \ln[\ln(z)] = \ln(L_1) \quad (51)$$

The derived authors' approximation (49) provides an excellent starting value for positive arguments and was subsequently refined using Halley's third-order iteration. In the present case, Eq. (49) gives the following result:

$$W_0\left(\beta Re^{\beta(d+k)}\right) = W_0(z) = 32.05957494264020$$

Thus, the proposed approximation already provides an excellent estimate of the principal branch W_0 , with a relative deviation of only 0.0000169% from the reference value.

It is emphasis to point out that the proposed logarithmic asymptotic approximation [Eq. (49)] for the principal branch W_0 represents a novel analytical contribution of the present work. To the best of the authors' knowledge, no comparable closed-form approximation combining such algebraic compactness, computational simplicity, and structural clarity has been reported in the literature. Expressed exclusively in terms of elementary logarithmic functions, the proposed approximation is straightforward to implement and provides an excellent initial estimate of W_0 . For the numerical example currently considered, the approximation yields a relative deviation of only 0.0000169% with respect to the reference value, indicating that the initial estimate is already remarkably close to the exact solution. Consequently, only a single Halley's third-order iteration is required to achieve essentially machine-precision accuracy for this example. Beyond its computational efficiency, the proposed approximation offers an elegant and transparent analytical representation of the principal branch of the Lambert W function, making it particularly attractive for explicit Lambert W -based formulations.

A particularly appealing feature of the proposed asymptotic approximation, Eq. (49), is the remarkable simplicity of its analytical structure. The expression is built exclusively from elementary logarithmic functions and a few small natural integer constants, such as 2 and 6, which arise directly from the asymptotic expansion. Consequently, the approximation is entirely free of empirically adjusted decimal coefficients or fitted numerical parameters. This feature not only enhances its mathematical elegance and transparency but also facilitates its implementation and highlights its rigorous analytical foundation.

After a single Halley refinement, the new value of W_0 is as follows:

$$W_0^{(1)}\left(\beta Re^{\beta(d+k)}\right) = W_0^{(1)}(z) = 32.059569523714697356$$

As can be seen, the relative deviation between the reference and the approximate computed values is only $4.37 \times 10^{-17}\%$, confirming that the proposed initial approximation, i.e., Eq. (49), followed by a single Halley iteration, yields essentially machine-precision accuracy.

According to Eq. (46), the fiction factor sought is as follows:

$$f = \left[\frac{1}{1.151292546497} \times 32.059569523714697342 \right]^{-2} = (6.311108569872)^{-2} \quad (46)$$

Thus, the friction factor f is finally:

$$f = 0.025106645888$$

Applying an iterative process to the implicit Eq. (4a), the friction factor f is obtained as:

$$f_{CW} = 0.025106645888$$

Since the two values coincide to all computed decimal places, the relative deviation is effectively zero within the adopted numerical precision.

From a mathematical perspective, this agreement is not coincidental but rather an intrinsic consequence of the derivation. The proposed Lambert W -based formulation is not an empirical approximation of the Colebrook-White equation; instead, it is an exact analytical transformation of the original implicit relationship. Therefore, when the Lambert W function is evaluated to machine precision (or to any desired numerical accuracy), the computed friction factor is rigorously identical to that obtained by solving the Colebrook-White equation iteratively. Any residual difference reported in numerical comparisons can only be attributed to finite-precision arithmetic, truncation, or rounding errors associated with the numerical evaluation of the Lambert W function or the iterative solver. This exact equivalence represents a major advantage of the proposed formulation, as it preserves the full mathematical fidelity of the Colebrook-White equation while providing a fully explicit analytical expression for the friction factor.

To illustrate the practical performance of the proposed logarithmic asymptotic approximation, Eq. (49), for the principal branch W_0 , a comparative numerical study was carried out using the same relative roughness as that adopted in the illustrative example ($r = 0.002$). The Reynolds number was varied throughout the turbulent-flow regime, and the Darcy friction factor was evaluated using the proposed Colebrook-Lambert formulation, Eq. (46), with both the approximate and the reference values of the Lambert W function. The purpose of this comparison is to assess the influence of the proposed approximation on the computed friction factor over a representative range of Reynolds numbers. The results, summarized in Table 1, demonstrate the practical accuracy of the proposed approximation while providing an indication of its suitability for engineering applications.

Table 1: Comparison of the Colebrook-Lambert formulation using approximate and exact W_0 , for the relative roughness $r = 0.002$.

Relative roughness $r = 0.002$			
Reynolds number R	f Eq. (46) using approximate W_0 Eq. (49)	f Eq. (46) using exact W_0	Deviation (%)
4,000	0.041888398743	0.041890912817	6.001477E-03
5,000	0.039563714962	0.039566021366	5.829252E-03
10,000	0.033792229135	0.033793731358	4.445272E-03

20,000	0.029787589131	0.029788346197	2.541487E-03
50,000	0.026505410587	0.026505591909	6.840889E-04
100,000	0.025106608439	0.025106645888	1.491596E-04
200,000	0.024309337974	0.024309342713	1.949466E-05
500,000	0.023788840909	0.023788840996	3.663484E-07
1,000,000	0.023606990413	0.023606990398	6.237341E-08
2,000,000	0.023514343334	0.023514343330	1.724857E-08
5,000,000	0.023458181504	0.023458181503	1.371875E-09
10,000,000	0.023439363312	0.023439363312	1.609547E-10

For this illustrative test case, the friction factors obtained with the approximate W_0 are virtually indistinguishable from those computed with the refined W_0 . The deviations remain extremely small throughout the selected turbulent range, confirming that the proposed initial approximation, Eq. (49), is already sufficiently accurate for direct use in the Colebrook-Lambert formulation, Eq. (46), for the present example. A broader parametric study would nevertheless be required before generalizing this conclusion to all roughness and Reynolds-number combinations.

As shown in Table 1, for the fixed relative roughness r , the relative deviation between the friction factors computed with the approximate and reference values of W_0 decreases progressively as the Reynolds number increases. This behavior is consistent with the asymptotic nature of the proposed approximation. Since the argument of the Lambert W function increases with the Reynolds number R , the logarithmic asymptotic approximation becomes increasingly accurate for larger values of R . Consequently, the largest deviations are naturally observed at the lower end of the turbulent regime, where the argument of W_0 is comparatively smaller. Even in this less favorable region, however, the deviations remain extremely small, confirming the excellent accuracy and robustness of the proposed approximation over the entire range of Reynolds numbers considered.

The results presented In Table 2 provide further evidence of the reliability and robustness of the proposed approximation for the principal branch W_0 [Eq. (49)] when applied to a relative roughness of $r = 0.005$. Using the same computational procedure, the friction factors obtained with the approximate and reference values of the Lambert W function remain in excellent agreement over the entire turbulent Reynolds number range investigated. The uniformly negligible relative deviations confirm that Eq. (49) constitutes a highly reliable initial approximation of W_0 , capable of reproducing the reference Colebrook-Lambert solution with excellent accuracy. These findings reinforce the practical value of the proposed approximation and demonstrate its suitability for high-accuracy engineering and scientific applications.

Table 2: Comparison of the Colebrook-Lambert formulation using approximate and exact W_0 , for the relative roughness $r = 0.005$.

Relative roughness $r = 0.005$			
Reynolds number R	f Eq. (46) using approximate W_0 Eq. (49)	f Eq. (46) using exact W_0	Deviation (%)
4,000	0.044708671756	0.044711236317	5.735832E-03
5,000	0.042610699054	0.042612847789	5.042456E-03
10,000	0.037628878338	0.037629861976	2.613981E-03
20,000	0.034469739508	0.034470044152	8.837921E-04
50,000	0.032174612603	0.032174645220	1.013750E-04
100,000	0.031306379581	0.031306382937	1.071978E-05
200,000	0.030846766806	0.030846766941	4.374770E-07
500,000	0.030561706557	0.030561706540	5.360127E-08
1,000,000	0.030465025824	0.030465025821	1.117093E-08
2,000,000	0.030416362596	0.030416362596	1.565594E-09
5,000,000	0.030387059799	0.030387059799	8.977153E-11
10,000,000	0.030377274593	0.030377274593	9.216952E-12

To further assess the reliability and robustness of the proposed asymptotic approximation for the principal branch W_0 , an additional comparative study was conducted for the limiting case of a hydraulically smooth pipe, corresponding to $r \rightarrow 0$. This limiting condition is of particular theoretical and practical interest, as it represents one of the two fundamental asymptotic regimes of the Colebrook-White equation. Using the same computational procedure adopted in the previous comparisons, the Reynolds number R was varied throughout the turbulent-flow regime, and the Darcy friction factor was evaluated using the proposed Colebrook-Lambert formulation with both the approximate and the reference values of the Lambert W function. The primary objective of this analysis is to examine whether the proposed approximation preserves its accuracy under smooth-wall conditions and to evaluate its influence on the computed friction factor over a broad range of turbulent Reynolds numbers. The corresponding results are summarized in Table.3.

To complete the numerical assessment of the proposed asymptotic approximation for the principal branch W_0 , Eq. (49), a final comparative study was performed for a relative roughness of $r = 0.037$, corresponding to the upper limiting curve of the Moody diagram and representing the case of highly rough pipes. This limiting condition is of particular engineering significance because it defines one of the extreme operating regimes encountered in turbulent pipe-flow analysis. Following the same computational procedure adopted in the previous investigations, the Reynolds number R was varied over a representative turbulent-flow range, and the Darcy friction factor was evaluated using the proposed Colebrook-Lambert formulation with both the approximate and the reference values of the Lambert W function. The objective of this final comparison is to assess the robustness and numerical reliability of the proposed approximation under severe roughness conditions and to determine its influence on the computed friction factor in one

of the most demanding practical situations. The corresponding results are presented in Table 4.

Table 3: Comparison of the Colebrook-Lambert formulation using approximate and exact W_0 , for the relative roughness $r \rightarrow 0$.

Relative roughness $r \rightarrow 0$			
Reynolds number R	f Eq. (46) using approximate W_0 Eq. (49)	f Eq. (46) using exact W_0	Deviation (%)
4,000	0.039905057022	0.039907014056	4.903985E-03
5,000	0.037390814974	0.037392727578	5.114910E-03
10,000	0.030881364090	0.030882950353	5.136372E-03
20,000	0.025881865447	0.025883078538	4.686812E-03
50,000	0.020890628555	0.020891443528	3.900991E-03
100,000	0.017989175332	0.017989773084	3.322733E-03
200,000	0.015636786038	0.015637225006	2.807202E-03
500,000	0.013157652397	0.013157946657	2.236372E-03
1,000,000	0.011644821706	0.011645040998	1.883135E-03
2,000,000	0.010372725282	0.010372890051	1.588459E-03
5,000,000	0.008981125382	0.008981239776	1.273705E-03
10,000,000	0.008102581777	0.008102669431	1.081788E-03

Table 4: Comparison of the Colebrook-Lambert formulation using approximate and exact W_0 , for the relative roughness $r = 0.037$.

Relative roughness $r = 0.037$			
Reynolds number R	f Eq. (46) using approximate W_0 Eq. (49)	f Eq. (46) using exact W_0	Deviation (%)
4,000	0.068756994362	0.068757482402	7.097985E-04
5,000	0.067582459983	0.067582732874	4.037893E-04
10,000	0.065126360195	0.065126393122	5.055817E-05
20,000	0.063836909895	0.063836912085	3.429518E-06
50,000	0.063040827464	0.063040827397	1.059982E-07
100,000	0.062771457889	0.062771457868	3.450345E-08
200,000	0.062635993290	0.062635993287	5.462451E-09
500,000	0.062554461192	0.062554461191	3.380340E-10
1,000,000	0.062527241278	0.062527241278	3.592362E-11
2,000,000	0.062513623313	0.062513623313	3.544690E-12
5,000,000	0.062505449967	0.062505449967	1.531323E-13
10,000,000	0.062502725091	0.062502725091	1.358525E-14

As evidenced by Table 3, the friction factors computed using the proposed approximation of W_0 , Eq. (49), remain in excellent agreement with those obtained from the reference Lambert W function throughout the entire Reynolds number R range investigated. This comparison demonstrates that the proposed approximation retains its high level of accuracy even in the limiting case of a hydraulically smooth pipe, thereby confirming its robustness and broad applicability. Together with the previous results obtained for rough pipes, these findings provide strong evidence that the proposed approximation constitutes a reliable and efficient alternative for evaluating the Lambert W -based explicit Colebrook formulation over a wide range of engineering conditions.

The results reported in Table 4 clearly demonstrate that the proposed approximation retains an excellent level of accuracy even for highly rough pipes. Throughout the turbulent Reynolds number R range considered, the friction factors computed using the approximate and reference values of the principal branch W_0 remain in excellent agreement, with only negligible relative deviations. As observed in the previous numerical studies, the relative deviation decreases progressively as the Reynolds number increases. This behavior is fully consistent with the asymptotic nature of the proposed approximation, since the argument of the Lambert W function increases with the Reynolds number, thereby improving the accuracy of the logarithmic asymptotic expansion. Consequently, the largest deviations are naturally observed at the lowest Reynolds numbers considered, whereas the approximation becomes increasingly accurate as the flow approaches the fully turbulent regime. These results provide further evidence that Eq. (49) constitutes a robust and reliable approximation of the principal branch W_0 , remaining accurate over the entire spectrum of practical roughness conditions, from hydraulically smooth pipes to the upper roughness limit represented by the Moody diagram.

It should be emphasized that the deviations reported in Tables 1-4 correspond solely to the proposed explicit approximation of W_0 , Eq. (49), without any iterative correction. Whenever greater accuracy is required, a single Halley refinement is sufficient to produce results that are virtually indistinguishable from the reference Lambert W solution, yielding accuracy at essentially the level of machine precision. This dual capability makes the proposed approximation particularly attractive, as it can be employed either as a simple explicit formula or as an exceptionally efficient initial guess requiring only one iteration to achieve near-exact results.

ASYMPTOTIC CONSISTENCY OF THE PROPOSED W_0 APPROXIMATION

To verify the asymptotic consistency of the approximate Eq. (49), let's $z \rightarrow \infty$. Since,

$$L_1 = \ln(z) \rightarrow \infty,$$

whereas

$$L_2 = \ln(L_1)$$

increases much more slowly than L_1 , it follows that:

$$\frac{L_2}{L_1} \rightarrow 0$$

Thus, the following quantity is such as

$$\frac{L_2(L_2 - 2)}{2L_1^2} = \frac{1}{2} \left(\frac{L_2}{L_1} \right) \left(\frac{L_2}{L_1} - \frac{2}{L_1} \right) \rightarrow 0$$

and

$$\frac{L_2(2L_2^2 - 9L_2 + 6)}{6L_1^3} = \frac{1}{6} \left(\frac{L_2}{L_1} \right) \left(\frac{2L_2^2}{L_1^2} - \frac{9L_2}{L_1^2} + \frac{6}{L_1^2} \right) \rightarrow 0$$

Consequently, all higher-order correction terms vanish in the limit, leaving only the following two dominant terms in Eq. (49):

$$W_0(z) \approx L_1 - L_2$$

Substituting the definitions of L_1 and L_2 , expressed by Eqs. (50) and (51), respectively, yields the following final result:

$$w_0(z) \approx \ln(z) - \ln[\ln(z)], \quad z \rightarrow \infty$$

Eq. (47) is then reproduced.

Therefore, the proposed approximation, Eq. (49), is fully consistent with the classical asymptotic expansion of the principal branch of the Lambert W function, valid for large values of the argument z , expressed by Eq. (47). The additional terms constitute higher-order corrections that vanish as the argument increases, thereby improving the approximation for finite values of the argument z while preserving the correct asymptotic behavior. This result provides an important theoretical validation of the proposed approximation, demonstrating that it not only offers a highly accurate finite-argument representation but also recovers the well-established asymptotic expression of the Lambert W function in the limit of large arguments.

ENERGETIC BRANCH FORMULATION OF THE LAMINAR-TURBULENT FRICTION LAW

The explicit Lambert W -based formulation developed in the previous section provides an exact analytical solution of the Colebrook-White equation and is therefore restricted to the turbulent flow regime. A complete hydraulic formulation, however, should also

account for the exact laminar solution while preserving the analytical consistency of both limiting regimes.

Most existing unified friction-factor equations address this problem by introducing empirical transition functions, blending parameters, or critical Reynolds numbers. Although these approaches generally provide satisfactory numerical predictions, they inevitably rely on additional empirical coefficients whose physical interpretation is often uncertain.

The philosophy adopted in the present work is fundamentally different. Rather than constructing an empirical interpolation between laminar and turbulent flows, the proposed formulation treats the two regimes as two exact dissipative branches of the same hydraulic problem. The laminar branch is represented by the classical Hagen-Poiseuille solution, whereas the turbulent branch is represented by the exact Colebrook-Lambert solution derived in the previous section. A unified explicit friction law is then obtained through an energetic branch-selection criterion that preserves the exact analytical structure of both limiting solutions without introducing any additional empirical parameters.

This approach provides a simple, physically interpretable, and mathematically consistent framework for unifying the laminar and turbulent friction laws while maintaining the exactness of their respective governing equations.

Thus, the objective is to construct a single expression for the Darcy-Weisbach friction factor f which satisfies the two exact limiting requirements.

There exist an infinitely many continuous functions satisfying the exact laminar and turbulent asymptotic limits, the Lehmer mean (Lehmer, 1971; Bullen, 2003) provides a particularly simple analytical interpolation. It introduces no empirical parameters, exactly recovers both asymptotic regimes, is infinitely differentiable, symmetric, homogeneous of degree one, and therefore constitutes a mathematically attractive candidate for a unified friction law.

However, in this section, the proposed unified law should not be interpreted as an empirical smoothing of the laminar-turbulent transition. Rather, it is an energetic branch formulation in which the laminar solution and the exact Colebrook-Lambert turbulent solution are treated as two admissible dissipative branches. The resulting upper-envelope expression introduces no transition parameter and preserves the exactness of both limiting regimes.

The previous sections established an exact analytical reformulation of the Colebrook-White equation and derived its explicit solution in terms of the Lambert W function. The resulting expression is mathematically equivalent to the original implicit equation and is therefore valid throughout the turbulent regime.

The remaining challenge consists in extending the formulation to include the laminar regime while preserving the exact asymptotic behavior of both flow regimes.

For fully developed laminar flow in a circular pipe, the Darcy friction factor f_L , where the subscript L denotes “Laminar”, is given exactly by the following:

$$f_L = \frac{64}{R} \quad (52)$$

whereas the turbulent friction factor f_T , where the subscript T denotes “Turbulent”, is represented by the explicit Colebrook-Lambert solution, expressed by Eq. (46).

For a given pipe geometry and flow velocity, the quantities L , D , V , and g in Eq. (1) remain fixed. Consequently, one may write what follows:

$$\Delta h \propto f \quad (53)$$

This shows that the Darcy friction factor is directly proportional to the irreversible energy dissipation associated with the flow. Accordingly, the laminar and turbulent solutions may be interpreted as two distinct admissible dissipative branches, each corresponding to a different flow structure. Thus, the following can be written:

$$f \in \{f_L, f_T\} \quad (54)$$

Rather than introducing an empirical blending function or a transition parameter, the present work adopts a branch-selection approach. The physically admissible friction factor is assumed to coincide with the branch producing the largest energy dissipation. This assumption leads to the following unified expression:

$$f_U = \max(f_L, f_T) \quad (55)$$

The subscript U denotes “Unified”.

Eq. (55) introduces neither a critical Reynolds number nor an empirical transition exponent. Moreover, it preserves exactly both asymptotic solutions.

For numerical implementation, the maximum operator may be expressed analytically through the following identity:

$$\max(a, b) = \frac{a + b + |a - b|}{2} \quad (56)$$

Applying Eq. (56) to the laminar and turbulent friction factors yields the following:

$$f_U = \frac{f_L + f_T + |f_L - f_T|}{2} \quad (57)$$

As overmentioned, Eq. (57) preserves both asymptotic solutions.

Indeed, if:

$$f_L \gg f_T \quad (58)$$

then Eq. (57) gives the following:

$$f_U \approx \frac{2f_L}{2} = f_L = \frac{64}{R} \quad (59)$$

and the exact laminar solution is recovered.

Conversely, if:

$$f_T \gg f_L \quad (60)$$

then Eq. (57) allows writing the following:

$$f_U \approx \frac{2f_T}{2} = f_T \quad (61)$$

which coincides exactly with the Colebrook-Lambert solution, expressed by Eq. (46).

Substituting Eqs. (46) and (52) into Eq. (57) gives the following explicit unified formulation:

$$f_U = \frac{1}{2} \left\{ \frac{64}{R} + \left[\beta^{-1} W \left(\beta Re^{\beta(d+k)} \right) - crR \right]^{-2} + \left| \frac{64}{R} - \left[\beta^{-1} W \left(\beta Re^{\beta(d+k)} \right) - crR \right]^{-2} \right| \right\} \quad (62)$$

The proposed formulation possesses several attractive properties.

First, it preserves the exact analytical solution in the laminar regime, such as writing the following:

$$f_U \rightarrow \frac{64}{R} \quad (63)$$

Second, it recovers exactly the Colebrook-White equation in the turbulent regime, since the following can be written:

$$f_U \rightarrow f_T = \left[\beta^{-1} W \left(\beta Re^{\beta(d+k)} \right) - crR \right]^{-2} \quad (64)$$

Third, no additional empirical coefficient, transition exponent, or critical Reynolds number is introduced. The unified expression therefore retains the exact mathematical structure of both limiting solutions while remaining fully explicit.

It should be emphasized that the proposed formulation, i.e., Eq. (62), is not intended to represent a continuous empirical interpolation through the transition region. Instead, it should be interpreted as an energetic branch formulation, in which the laminar solution

and the exact Colebrook-Lambert solution constitute two admissible dissipative branches of the same hydraulic problem. The unified friction factor is obtained by selecting the branch associated with the larger energy dissipation, thereby preserving the exact analytical description of both asymptotic flow regimes without introducing any additional empirical parameters.

Furthermore, the analytical framework established herein extends beyond the explicit inversion itself. It naturally supports the development of a unified dual-branch friction formulation that incorporates both the exact Hagen-Poiseuille law and the exact Colebrook-Lambert solution without introducing empirical blending functions or additional fitting parameters. In this sense, the Lambert W solution becomes one component of a broader analytical framework rather than the final objective of the derivation.

For these reasons, the present work should not be viewed simply as another Lambert W -based solution of the Colebrook-White equation. Instead, it establishes a new analytical framework in which the exact reformulation, the exact explicit inversion, and the unified friction formulation constitute three logically connected steps of the same theoretical development.

EXPLICIT LAMBERT W -BASED SOLUTION OF THE PRANDTL-VON KÁRMÁN FRICTION FACTOR

Objective

The Colebrook equation is not the only implicit hydraulic relationship that can be transformed into an explicit Lambert W -based formulation. Several classical resistance laws governing turbulent pipe flow possess a mathematical structure that makes them suitable for analytical inversion through the Lambert W function. Among these, the well-known Prandtl-von Kármán equation for hydraulically smooth pipes represents a particularly important case because of its widespread use in hydraulic engineering and fluid mechanics.

Motivated by the excellent performance of the proposed asymptotic approximation for the principal branch W_0 , Eq. (49), the present section extends its application beyond the Colebrook equation. The objective is twofold. First, an exact explicit solution of the Prandtl-von Kármán equation is derived by transforming the original implicit equation into the canonical Lambert W form. Second, the newly proposed approximation of W_0 is incorporated into the resulting solution to obtain a fully explicit analytical expression requiring only elementary logarithmic evaluations. This example demonstrates that the proposed approximation is not restricted to the Colebrook-Lambert formulation but can also be successfully applied to other implicit nonlinear equations encountered in hydraulics.

Beyond illustrating the versatility of the proposed approximation, this development provides an independent validation of its analytical and numerical capabilities. The proposed explicit formulation is validated through a direct comparison between the friction factors obtained from the original implicit Prandtl-von Kármán equation and those predicted by the proposed Lambert W -based approximation, Eq. (49). This comparison provides an objective assessment of the accuracy, robustness, and computational efficiency of the proposed approach.

Consequently, this example broadens the scope of the present work and highlights the potential of the proposed W_0 approximation as a general analytical tool for solving implicit equations arising in hydraulic engineering.

Explicit Prandtl-von Kármán friction factor equation

Let's start from the Prandtl-von Kármán smooth-pipe equation, recalled as follows (von Kármán, 1930; Prandtl, 1935; White and Xue, 2021):

$$\frac{1}{\sqrt{f}} = 2 \log_{10} \left(R \sqrt{f} \right) - \frac{4}{5} \quad (65)$$

Introducing the following auxiliary variable

$$y = \frac{1}{\sqrt{f}} \quad (66)$$

results in what follows:

$$\sqrt{f} = \frac{1}{y} \quad (67)$$

Substituting Eqs. (66) and (67) into Eq. (65) yields the following:

$$y = 2 \log_{10} \left(\frac{R}{y} \right) - \frac{4}{5} \quad (68)$$

Using the following identity

$$\log_{10}(u) = \frac{\ln(u)}{\ln(10)} \quad (69)$$

Eq. (68) becomes as follows:

$$y = \frac{2}{\ln(10)} \ln \left(\frac{R}{y} \right) - \frac{4}{5} \quad (70)$$

After expanding the logarithm, Eq. (70) can be rewritten in the following form:

$$y = \frac{2}{\ln(10)} [\ln(R) - \ln(y)] - \frac{4}{5} \quad (71)$$

Let's adopt the following:

$$a = \frac{2}{\ln(10)}, b = -\frac{4}{5} \quad (72)$$

Then Eq. (71) may be written as follows:

$$y = a \ln(R) - a \ln(y) + b \quad (73)$$

Rearranging all terms involving y on the left-hand side yields the following:

$$y + a \ln(y) = a \ln(R) + b \quad (74)$$

Dividing Eq. (74) by a , one obtains what follows:

$$\frac{y}{a} + \ln(y) = \ln(R) + \frac{b}{a} \quad (75)$$

Now write the following:

$$y = aY \quad (76)$$

This allows us writing what follows:

$$\ln(y) = \ln(aY) = \ln(a) + \ln(Y) \quad (77)$$

Substituting Eqs. (76) and (77) into Eq. (75) results in the following:

$$Y + \ln(a) + \ln(Y) = \ln(R) + \frac{b}{a} \quad (78)$$

Combining the logarithmic terms, Eq. (78) becomes as follows:

$$Y + \ln(Y) = \ln(R) - \ln(a) + \frac{b}{a} \quad (79)$$

Eq. (79) can be rewritten in the following form:

$$Y + \ln(Y) = \ln\left(\frac{R}{a}\right) + \frac{b}{a} \quad (80)$$

Exponentiating both sides of Eq. (80), we obtain the following:

$$e^{Y + \ln(Y)} = e^{\ln(R/a) + \frac{b}{a}} \quad (81)$$

The left-hand side becomes as follows:

$$e^{Y + \ln(Y)} = Y e^Y \quad (82)$$

while the right-hand side can be written as follows:

$$e^{\ln(R/a) + \frac{b}{a}} = \frac{R}{a} e^{b/a} \quad (83)$$

Combining Eqs. (81), (82), and (83) results in the following:

$$Y e^Y = \frac{R}{a} e^{b/a} \quad (84)$$

By definition of the principal branch of the Lambert function, Eq. (84) allows us writing the following:

$$Y = W_0\left(\frac{R}{a} e^{b/a}\right) \quad (85)$$

Using Eq. (76), the solution for y is therefore as follows:

$$y = a W_0\left(\frac{R}{a} e^{b/a}\right) = a W_0(z) \quad (86)$$

Substituting Eq. (86) into Eq. (67) yields the following final result:

$$f = \left[a W_0\left(\frac{R}{a} e^{b/a}\right) \right]^{-2} \quad (87)$$

Let's recall that the constants a and b are defined by Eq. (72).

Thus, the originally implicit Prandtl-von Kármán equation is transformed into a fully explicit expression involving only the principal branch W_0 of the Lambert function.

Substituting the approximate Eq. (49) into Eq. (87) yields the following explicit Prandtl-von Kármán friction factor:

$$f \approx \left[a \left(L_1 - L_2 + \frac{L_2}{L_1} + \frac{L_2(L_2 - 2)}{2L_1^2} + \frac{L_2(2L_2^2 - 9L_2 + 6)}{6L_1^3} \right) \right]^{-2} \quad (88)$$

Where L_1 and L_2 are defined by Eqs (50) and (51), respectively, as a function of the argument z defined by Eq. (86).

This result is important because it shows that the proposed approximation of W_0 is not restricted to the Colebrook-Lambert formulation. It also provides a direct and efficient way to solve other implicit hydraulic resistance laws, such as the Prandtl-von Kármán smooth-pipe equation. The resulting formula is explicit, compact, and computationally simple, while retaining high numerical accuracy.

To evaluate the practical performance of the explicit approximation given by Eq. (88), a comparative numerical study was carried out over a representative range of turbulent Reynolds numbers R . The Darcy friction factor was computed in two different ways: first, by solving the original implicit Prandtl-von Kármán Eq. (65), which provides the reference solution, and second, by directly applying the proposed explicit Lambert W -based approximation [Eq. (88)]. The objective of this comparison is to quantify the accuracy of the proposed explicit formulation and to assess its capability to reproduce the original implicit solution without requiring any iterative procedure. The corresponding friction factors and their relative deviations are summarized in Table 5.

Table 5: Validation of the proposed explicit Lambert W -based approximation [Eq. (88)] through comparison with the original implicit Prandtl-von Kármán Eq. (65)

Reynolds number R	Exact friction factor f original Eq. (65)	Approx. friction factor f final Eq. (88)	Deviation (%)
4,000	0.039915881576132274	0.039913924487867417	0.004903031544
10,000	0.030889096376883460	0.030887509700161755	0.005136688695
50,000	0.020894945325178691	0.020894130079349225	0.003901641362
100,000	0.017992593917693433	0.017991995964475913	0.003323329700
500,000	0.013159738192800184	0.013159443837135536	0.002236789671
1,000,000	0.011646540648628143	0.011646321287889018	0.001883484081
5,000,000	0.008982266220230651	0.008982151792182303	0.001273932942
10,000,000	0.008103552371798209	0.008103464693168784	0.001081977698
50,000,000	0.006492777266892385	0.006492728525079086	0.000750708230
100,000,000	0.005941026453368196	0.005940988119529114	0.000645239327

As shown in Table 5, the friction factors predicted by Eq. (88) are in excellent agreement with those obtained from the original implicit Prandtl-von Kármán equation over the entire Reynolds number range investigated. The maximum deviation is only approximately 0.0051%, occurring at a Reynolds number of $R = 10,000$. As the Reynolds number increases, the deviation decreases steadily, reflecting the asymptotic nature of the proposed approximation and its progressively improved accuracy for larger values of the Lambert W function argument.

The consistently small relative deviations demonstrate that the proposed approximation of the principal branch W_0 , expressed by Eq. (88), accurately preserves the nonlinear characteristics of the original equation while completely eliminating the need for iterative computations. These results confirm that Eq. (88) provides a reliable, compact, and computationally efficient explicit alternative for evaluating the Darcy friction factor in hydraulically smooth turbulent pipe flow.

EXTENSION TO GENERAL IMPLICIT LOGARITHMIC FRICTION LAWS

Presentation and objective

The applicability of the proposed approximation of the principal branch W_0 , Eq. (49), is not limited to pipe-flow resistance equations. A broader class of implicit logarithmic friction laws encountered in hydraulics and turbulent-flow analysis can also be transformed into the canonical Lambert W form. Among these, turbulent skin-friction correlations for hydraulically smooth surfaces provide a particularly relevant example, since the unknown friction coefficient appears simultaneously as an algebraic term and inside a logarithmic function. These equations often exhibit the same mathematical structure as the Prandtl-von Kármán smooth-pipe relation considered previously.

The analytical procedure developed herein is not restricted to the Colebrook-White and Prandtl-von Kármán equations. More generally, any implicit hydraulic relation that can be transformed into the following canonical form:

$$Y + \ln(Y) = C \quad (89)$$

or equivalently:

$$Ye^Y = X, \quad (90)$$

may be solved explicitly using the Lambert W function. Consequently, the proposed asymptotic approximation of the principal branch W_0 , Eq. (49), has the potential to serve as a general analytical tool for deriving compact explicit solutions to a broader class of logarithmic hydraulic equations.

The generalized logarithmic resistance law encompasses several classical hydraulic resistance equations, including the open-channel formulations developed by Keulegan (1938) and subsequently presented in standard hydraulic textbooks such as Chow (1959).

Although Keulegan-type logarithmic resistance laws may also be transformed into the Lambert W form, their mathematical structure is essentially analogous to that of the Prandtl-von Kármán equation after introducing a rescaled Reynolds number. Therefore, they should be regarded as members of the same class of implicit logarithmic friction laws rather than as a fundamentally distinct application.

The objective of this section is to show, in a detailed and rigorous manner, that such equations defined by (89) can be analytically inverted using the Lambert W function and that the proposed approximation of W_0 can therefore be used to construct compact, explicit, and computationally efficient solutions. This further confirms the potential of the proposed approximation as a general analytical tool for treating implicit logarithmic friction laws in hydraulics and turbulent-flow analysis.

Detailed derivation

Let's consider a general implicit logarithmic friction law of the following form:

$$\frac{1}{\sqrt{C_f}} = A \log_{10}(R \sqrt{C_f}) + B \quad (91)$$

The constants A and B are employed herein as generic parameters to illustrate that a broad family of implicit logarithmic equations can be handled using an identical analytical procedure. In practical applications, however, these constants are prescribed by the particular physical correlation being considered and are therefore known in advance.

For example, in the Prandtl-von Kármán equation applicable to hydraulically smooth pipes, which was examined in the preceding section, the constants are $A = 2$ and $B = -4/5$. Likewise, in the von Kármán logarithmic skin-friction law for smooth flat plates, the values of these constants depend on the particular formulation and the definition adopted for the skin-friction coefficient C_f ; a commonly used form is characterized by $A = 4$ and $B = -2/5$. Therefore, A and B are not adjustable parameters but fixed constants prescribed by the specific physical correlation under consideration.

Introducing the following auxiliary variable:

$$y = \frac{1}{\sqrt{C_f}} \quad (92)$$

results in what follows:

$$\sqrt{C_f} = \frac{1}{y} \quad (93)$$

Substituting Eqs. (92) and (93) into Eq. (91) yields the following:

$$y = A \log_{10}\left(\frac{R}{y}\right) + B \quad (94)$$

Using the following identity

$$\log_{10}(u) = \frac{\ln(u)}{\ln(10)} \quad (95)$$

Eq. (94) becomes as follows:

$$y = \frac{A}{\ln(10)} \ln\left(\frac{R}{y}\right) + B \quad (96)$$

After expanding the logarithm, Eq. (96) can be rewritten in the following form:

$$y = \frac{A}{\ln(10)} [\ln(R) - \ln(y)] + B \quad (97)$$

Let's adopt the following:

$$a = \frac{A}{\ln(10)} \quad (98)$$

Then Eq. (97) may be written as follows:

$$y = a \ln(R) - a \ln(y) + B \quad (99)$$

Rearranging all terms involving y on the left-hand side yields the following:

$$y + a \ln(y) = a \ln(R) + B \quad (100)$$

Dividing Eq. (100) by a , one obtains what follows:

$$\frac{y}{a} + \ln(y) = \ln(R) + \frac{B}{a} \quad (101)$$

Now write the following:

$$y = aY \quad (102)$$

This allows us writing what follows:

$$\ln(y) = \ln(aY) = \ln(a) + \ln(Y) \quad (103)$$

Substituting Eqs. (102) and (103) into Eq. (101) results in the following:

$$Y + \ln(a) + \ln(Y) = \ln(R) + \frac{B}{a} \quad (104)$$

Combining the logarithmic terms, Eq. (104) becomes as follows:

$$Y + \ln(Y) = \ln(R) - \ln(a) + \frac{B}{a} \quad (105)$$

Eq. (105) can be rewritten in the following form:

$$Y + \ln(Y) = \ln\left(\frac{R}{a}\right) + \frac{B}{a} \quad (106)$$

Exponentiating both sides of Eq. (106), we obtain the following:

$$e^{Y + \ln(Y)} = e^{\ln(R/a) + \frac{B}{a}} \quad (107)$$

The left-hand side becomes as follows:

$$e^{Y + \ln(Y)} = Y e^Y \quad (108)$$

while the right-hand side can be written as follows:

$$e^{\ln(R/a) + \frac{B}{a}} = \frac{R}{a} e^{B/a} \quad (109)$$

Combining Eqs. (107), (108), and (109) results in the following:

$$Y e^Y = \frac{R}{a} e^{B/a} \quad (110)$$

By definition of the principal branch of the Lambert function, Eq. (110) allows us writing the following:

$$Y = W_0\left(\frac{R}{a} e^{B/a}\right) \quad (111)$$

Using Eq. (102), the solution for y is therefore as follows:

$$y = a W_0\left(\frac{R}{a} e^{B/a}\right) = a W_0(z) \quad (112)$$

Substituting Eq. (112) into Eq. (93) yields the following final result:

$$C_f = \left[a W_0\left(\frac{R}{a} e^{B/a}\right) \right]^{-2} \quad (113)$$

Let's recall that the constant a is defined by Eq. (98).

Thus, the general implicit logarithmic skin-friction law equation, expressed by Eq. (91), is transformed into a fully explicit expression involving only the principal branch W_0 of the Lambert function.

Substituting the approximate Eq. (49) into Eq. (113) yields the following explicit friction coefficient relationship:

$$C_f \approx \left[a \left(L_1 - L_2 + \frac{L_2}{L_1} + \frac{L_2(L_2 - 2)}{2L_1^2} + \frac{L_2(2L_2^2 - 9L_2 + 6)}{6L_1^3} \right) \right]^{-2} \quad (114)$$

Where L_1 and L_2 are defined by Eqs (50) and (51), respectively, as a function of the argument z defined by Eq. (112).

This derivation demonstrates that the proposed approximation of W_0 can be extended beyond the Colebrook-White relationship and Prandtl-von Kármán smooth pipe-flow equation. It applies naturally to a broader family of implicit logarithmic friction laws, provided that they can be reduced to the canonical structure defined by Eq. (89), or equivalently by Eq. (90). This confirms the versatility of the proposed approximation and strengthens its value as a general analytical tool for explicit modeling of turbulent-flow resistance.

CONCLUSION

The present study has introduced a new analytical framework for the determination of the Darcy friction factor by revisiting the classical Colebrook-White equation from a fundamentally different perspective. Rather than developing another explicit approximation, the analysis has demonstrated that the original implicit equation can be transformed exactly through elementary algebraic manipulations, thereby preserving its complete mathematical structure while considerably simplifying its analytical treatment.

The first major contribution of this work is the establishment of an exact analytical reformulation of the Colebrook-White equation into a compact logarithmic representation. This reformulation constitutes the cornerstone of the proposed methodology and provides a much simpler analytical structure than the original implicit equation.

Building upon this reformulation, the second contribution consists in deriving a fully explicit analytical solution by means of the Lambert W function. Unlike conventional explicit correlations, the proposed equation is not an empirical approximation but an exact analytical equivalent of the Colebrook-White equation. Consequently, the friction factors computed by the proposed formulation are mathematically identical to those obtained from the classical iterative solution, provided that the Lambert W function is evaluated with sufficient numerical precision. The detailed numerical example presented in this study confirms this exact equivalence, the relative deviation between the two solutions being effectively zero within the adopted numerical precision.

The third contribution extends the analytical framework beyond the turbulent regime through the introduction of a unified dual-branch formulation. By considering the exact Hagen-Poiseuille equation and the exact Colebrook-Lambert solution as two admissible

dissipative branches, the proposed formulation provides a unified explicit friction law without resorting to empirical blending functions, transition exponents, or prescribed critical Reynolds numbers. Consequently, the exact asymptotic behavior of both laminar and turbulent flow regimes is preserved while no additional empirical parameters are introduced.

It is worth emphasizing that the compact Lambert W -based solution developed in the present work stems directly from the proposed analytical reformulation of the Colebrook equation. By exposing the intrinsic logarithmic structure of the problem in a particularly transparent form, the new formulation provides a natural framework for analytical inversion and leads to an explicit solution of remarkable compactness. Thus, the proposed reformulation should not be regarded as a simple algebraic rearrangement, but rather as a significant analytical contribution that underpins the derivation of the new Colebrook–Lambert equation. Taken together, the new Colebrook formulation, the compact Lambert W -based explicit solution, and the novel asymptotic approximation of the principal branch W_0 constitute a coherent and complementary set of contributions, offering both mathematical elegance and computational efficiency for the analysis of turbulent pipe-flow friction.

From a broader perspective, the present work demonstrates that the implicit character of the Colebrook-White equation is not an intrinsic mathematical limitation but rather a consequence of its traditional formulation. Once the equation is expressed in an appropriate analytical form, its explicit inversion follows naturally through the Lambert W function. This observation provides a new insight into the mathematical structure of one of the most widely used equations in hydraulic engineering.

The methodology developed herein opens several promising perspectives for future research. The analytical reformulation proposed in this study may facilitate the development of new exact formulations for related hydraulic resistance equations and may also provide a convenient analytical basis for sensitivity analyses, uncertainty quantification, parameter estimation, optimization procedures, and symbolic mathematical developments.

Furthermore, the proposed dual-branch framework could be extended to investigate transitional flow regimes and generalized resistance laws applicable to non-circular conduits and open-channel flows.

Beyond providing a new explicit expression for the Darcy friction factor, the present work establishes a coherent analytical framework showing that exact reformulation, exact inversion, and unified hydraulic modelling can be achieved simultaneously without sacrificing either mathematical rigor or physical consistency. It is hoped that this framework will stimulate further developments toward exact analytical formulations of other classical problems in hydraulic engineering and fluid mechanics.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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APPENDIX

In the present appendix note, an explicit solution of the Colebrook-White equation (Eq. 4) is given in terms of a perturbation series expansion. The main advantage of such a solution is the simplicity with which any degree of accuracy could be reached without resorting to advanced mathematical special function or tedious trial-and-error process. For that let us the implicit Eq. (4):

$$\frac{1}{\sqrt{f}} = -2 \log_{10} \left(\frac{\varepsilon}{3.7D} + \frac{2.51}{R\sqrt{f}} \right) \quad (\text{A.1})$$

By the change of variables $X = 1/\sqrt{f}$, $A = 2.51/R$ and $B = \varepsilon/3.7D$, Eq. (A.1) is rewritten as follows:

$$X + 2 \log_{10}(B + AX) = 0 \quad (\text{A.2})$$

Now, introducing a small parameter σ in the transcendental Eq. (A2) as perturbation parameter, one can writes:

$$X + 2 \log_{10}(B + \sigma AX) = 0 \quad (\text{A.3})$$

When $\sigma = 0$, the solution is straightforward and reduces to that of the fully Rough turbulent flow regime when $R \rightarrow \infty$.

The explicit solution of Eq. (A3) is then formulated in terms of a series expansion of the perturbation parameter σ as follows:

$$X = X_0 + \sigma X_1 + \sigma^2 X_2 + \sigma^3 X_3 + \dots \quad (\text{A.4})$$

Or in compact form:

$$X = \sum_{n=0}^{\infty} \sigma^n X_n \quad (\text{A.5})$$

where X_n denotes the n -th order term of the perturbation series.

Substituting Eq. (A.5) into Eq (A.3), expanding in Taylor series and grouping terms of the same order in σ , allows sequentially the determination of X_n which are, for the first three, as follows:

$$n = 0 : X_0 = -2 \log_{10}(B) \quad (\text{A.6})$$

$$n = 1 : X_1 = \frac{4A \ln(B)}{B \ln(10)^2} \quad (\text{A.7})$$

$$n = 2 : X_2 = \frac{4A^2 \ln(B)}{B^2 \ln(10)^3} [\ln(B) - 2] \quad (\text{A.8})$$

and so on for higher orders. It is worth mentioning that the 0-th order solution corresponds to the non-perturbed solution, i.e. the von Karman equation.

Reporting results of Eqs. (A5) in Eq. (A4), setting $\sigma = 1$ and rearranging, the perturbation solution to the third-order is the expressed as follows:

$$X = \frac{2 \ln(B)}{3 \ln(10)^4 B^3} \left\{ \begin{aligned} &A^3 [8 \ln(B)^2 - 36 \ln(B) + 24] + \\ &A^2 [6 \ln(B) \ln(10) B - 12 \ln(10) B] + \\ &A [6 \ln(10)^2 B^2] - 3 \ln(10)^3 B^3 \end{aligned} \right\} \quad (\text{A.9})$$

For the numerical example treated before, i.e.

$$\varepsilon/D = 0.002 \text{ and } R = 10^5,$$

one obtains $X = 6.308052749$, which corresponds to $f = 0.02513097669$. The relative error is then 0.09% of the exact value, which is fairly acceptable for practical engineering applications. A higher degree of accuracy could be achieved by increasing the order of the perturbation series expansion, but from a practical viewpoint, Eq. (A.9) forms a good trade-off between accuracy and computational effort.