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A NOVEL RIGOROUS EXPLICIT FRAMEWORK FOR CRITICAL FLOW DEPTH IN TRAPEZOIDAL OPEN CHANNELS

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ABSTRACT

The determination of critical flow depth in trapezoidal open channels remains a classical yet inherently implicit problem in hydraulic engineering, as the governing relationship is strongly nonlinear and does not admit a closed-form solution in terms of elementary functions. This study develops a rigorous, unified, and practically implementable framework that overcomes this limitation by combining exact analysis with highly accurate explicit formulations. Starting from the classical critical-flow condition and trapezoidal geometry, a compact dimensionless governing equation is derived using physically meaningful parameters, namely a shape parameter and a normalized discharge parameter. Through an appropriate change of variables, the problem is recast into a canonical algebraic form equivalent to a trinomial sextic equation with a unique physically admissible root, thereby establishing a rigorous exact foundation.

On this basis, two complementary solution strategies are proposed. First, a quasi-exact computational approach is developed using a normalized formulation combined with a highly efficient initial estimate and a one-shot Newton iteration, yielding a maximum deviation strictly below 3.6×10^{-5} % over the entire admissible range. Second, a fully explicit analytical formulation is derived through a Transformed Asymptotic-Taylor Reconstruction Method, producing a closed-form expression for the governing parameter with a maximum deviation of only 7.1×10^{-6} %. The proposed formulation preserves the correct asymptotic behavior and rigorously recovers the classical rectangular and triangular channel solutions as limiting cases, ensuring full physical consistency.

A key feature of the study is that the developed models remain valid for unsymmetrical trapezoidal channels through the introduction of an equivalent side-slope parameter, which constitutes an exact transformation rather than an approximation. The resulting formulations eliminate the need for iterative procedures, graphical methods, or empirical fits, while maintaining quasi-exact accuracy. The study thus provides a new generation of analytical tools that combine mathematical rigor, computational efficiency, and broad applicability for hydraulic design and analysis.

Keywords: Critical flow, Trapezoidal channel, Newton iteration, Transformed Asymptotic-Taylor Reconstruction Method, explicit model, Hydraulic design, Hydraulic analysis.

INTRODUCTION

Critical flow plays a fundamental role in open-channel hydraulics, as it establishes a unique correspondence between discharge and flow depth for a given cross-sectional geometry. Traditionally, the critical condition is defined either as the state at which the specific energy reaches its minimum for a prescribed discharge, or equivalently, when the discharge attains a maximum for a given specific energy. This condition coincides with a Froude number equal to unity (Chow, 1959; Hager, 1985). Such a definition provides the basis for determining the critical depth and for distinguishing between subcritical (tranquil) and supercritical (rapid) flow regimes (Hager, 2010; Achour and Nebbar, 2015; Achour and Amara, 2020a).

More recent developments have revisited the concept of critical flow by incorporating additional physical factors such as channel slope, boundary roughness, and fluid viscosity into a unified analytical framework. These approaches formulate an implicit dimensionless relationship by combining the critical-flow condition with generalized discharge expressions, thereby explicitly accounting for viscous effects through modified Reynolds-type parameters. Applications of this framework across smooth, transitional, and fully rough turbulent regimes reveal that critical flow is not solely dictated by channel geometry. Instead, it arises only when the bed slope exceeds a certain threshold. Below this limit, the flow remains strictly subcritical regardless of discharge, whereas above it, two distinct critical states may emerge: one corresponding to shallow depths and another to deeper conditions, each associated with different flow rates. Furthermore, boundary roughness increases the slope required to reach criticality, while viscous effects are most pronounced in transitional regimes. Graphical tools developed within this context enable practitioners to relate critical and normal depths across varying slopes, offering practical insight into flow-regime transitions. Independent validation using the specific-energy principle confirms both the predicted critical conditions and the occurrence of dual solutions at higher slopes, thereby supporting the robustness of the theoretical framework (Achour and Bedjaoui, 2006; Lakehal and Achour, 2017; Nebbar and Achour, 2018; Achour and Amara, 2020b; 2020c; 2020d; Sehtal and Achour, 2023; 2024).

Analytical solutions for critical depth are available for a limited number of simple geometries, including rectangular, triangular, and parabolic channels (Chow, 1959; Wong and Zhou, 2004; Achour and Khattaoui, 2008). However, for many practical cross-sections, the governing equations remain implicit, which has historically necessitated the use of graphical aids, iterative procedures, or trial-and-error methods, often with limited precision (Liu et al., 2012). Classical charts have been widely employed for circular and trapezoidal channels (Chow, 1959; Henderson, 1966; French, 1987), while numerous studies have proposed explicit formulations for a variety of channel shapes, including circular, trapezoidal, egg-shaped, and semi-elliptical sections (Swamee, 1993; 2005; Vatankhah and Easa, 2011; Li et al., 2012; Cheng et al., 2018; Shang et al., 2019).

In most traditional analyses, the determination of critical depth relies exclusively on the critical-flow condition, without explicitly incorporating the effects of slope, roughness, or viscosity. Nevertheless, complementary investigations have progressively addressed these influences in both open channels and conduits (Achour and Amara, 2020a; 2020b; Hachemi-Rachedi, 2021).

The computation of critical depth remains a central problem in hydraulic engineering practice, with applications ranging from backwater profile calculations to flow measurement and channel design (Chow, 1959). For non-standard geometries, particularly trapezoidal sections, the implicit nature of the governing equations necessitates iterative solution procedures, which can be computationally demanding and inconvenient in routine applications.

Beyond empirical curve-fitting approaches, several analytical techniques have been proposed to invert these implicit relationships. Wang (1998) introduced a direct formulation based on nested iteration. Swamee and Rathie (2005), using Lagrange inversion, derived an explicit series solution, although its convergence is relatively slow. Varandili et al. (2019) expressed the solution in terms of nested radicals, with accuracy depending on the depth of expansion. Alternative strategies based on asymptotic matching have also been developed to produce compact approximations for both normal and critical depths (Swamee, 1994; Vatankhah, 2013). More recently, Amara and Achour (2023) employed a δ-perturbation approach (Bender et al., 1989) to construct a globally convergent series representation of the critical depth, yielding accurate results without iterative computation. The practicality of such explicit formulations has been illustrated in real-world canal applications (Elhakeem, 2017).

The present study aims to establish a rigorous, unified, and practically implementable framework for determining the critical flow depth in trapezoidal open channels, addressing the well-known difficulty that the governing critical-flow relationship is inherently implicit, strongly nonlinear, and does not admit a closed-form solution in terms of elementary functions; to overcome this limitation, the authors adopt a structured and analytically consistent methodology that begins with the classical critical-flow condition combined with trapezoidal geometry to derive a compact dimensionless governing equation expressed in terms of physically meaningful parameters, namely a shape parameter characterizing the influence of the side slopes and a flow parameter representing a normalized discharge, thereby reducing the problem to a single nonlinear algebraic relation that is both general and physically transparent; recognizing the analytical complexity of this formulation, the authors introduce an appropriate change of variables that transforms the governing equation into a more tractable canonical form, revealing its equivalence to a trinomial sextic equation with a unique physically admissible root, whose existence and uniqueness are rigorously ensured; based on this exact foundation, two complementary solution strategies are developed, namely a quasi-exact computational approach relying on a normalized formulation combined with a highly efficient initial estimate and a one-shot Newton-type iteration, which achieves an exceptional maximum deviation strictly below 0.000036 % over the entire admissible range, and a fully explicit analytical formulation derived through the Transformed Asymptotic-Taylor Reconstruction Method, which combines asymptotic expansion, local Taylor correction, and inverse transformation to yield a closed-form expression of remarkable accuracy, with a maximum deviation of only 7.1×10^{-6} % across the whole range of interest; the methodology is particularly relevant because it bridges the gap between exact but impractical implicit formulations and empirical approximations lacking theoretical rigor, ensuring uniform accuracy while preserving the correct asymptotic behavior and physical consistency of the solution; a major strength of the study lies in demonstrating that the proposed models are not restricted to symmetrical trapezoidal sections but remain fully valid for unsymmetrical trapezoidal channels through the introduction of an equivalent side-slope parameter, this equivalence being exact and rigorously justified, thereby significantly extending the applicability of the results; in comparison with existing literature, the work introduces several substantial advances, including the explicit identification of the underlying algebraic structure of the problem, the formulation of a unified dimensionless framework, the development of quasi-exact and fully explicit solutions with extremely small and rigorously quantified deviations, the preservation of asymptotic consistency through the exact recovery of rectangular and triangular channel limits, and the elimination of graphical or trial-and-error procedures, thus providing analytical tools that are both mathematically robust and operationally efficient; overall, the study represents a significant contribution to open-channel hydraulics by delivering a new generation of critical-depth models that combine exact theoretical grounding, exceptionally high accuracy, computational simplicity, and broad applicability, thereby offering a reliable and modern alternative to conventional methods.

GEOMETRIC CONSIDERATIONS

For a trapezoidal channel of bed width b and side slope m, m horizontal to 1 vertical (Fig. 1), the critical state follows from the equality of mean velocity and wave celerity (Chow, 1959; Henderson, 1966); it reads as follows:

$$\frac{\alpha Q^2}{g\cos\theta}=\frac{A_c^3}{T_c} \tag{1}$$

with energy-correction factor $\alpha \approx 1$ in practice, Q the discharge, g the acceleration due to gravity, A_c the critical wetted area, and T_c the top width at critical flow (Fig. 1). The term $\cos(\theta)$ accounts for the inclination of the channel bed**,** since θ is the angle between the channel bottom (bed) and the horizontal. The cosine appears because the governing equations are often derived using forces or energy projected along directions, and $\cos(\theta)$ converts quantities between the actual sloped channel and an equivalent horizontal reference frame (Chow, 1959; Henderson, 1966)**.** In the vast majority of practical hydraulic applications, channel slopes are exceedingly small, typically ranging from 0.0001 to 0.01. Accordingly, $\cos(\theta) \approx 1$, and it is therefore customary in engineering practice to disregard this term, thereby simplifying the governing Eq. (1) without introducing any appreciable error.

For a trapezoidal channel, denoting the critical flow depth by *yc*, one may express the following relationship:

$$A_c = b\,y_c + m\,y_c^2 \tag{2}$$

$$T_c = b + 2m\,y_c \tag{3}$$

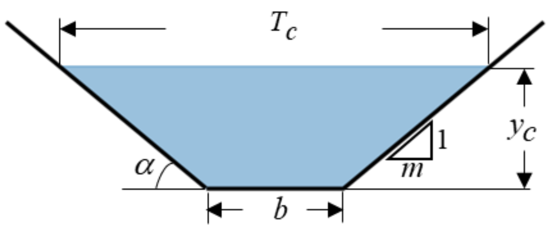

Figure 1: Schematic representation of a trapezoidal open channel at critical flow

DIMENSIONLESS PARAMETERS

The following dimensionless parameters are particularly meaningful, as they are grounded in physical considerations. These are the same dimensionless parameters used by the authors in an earlier study (Achour and Amara, 2025).

$$\eta = \frac{m\,y_c}{b} \tag{4}$$

$$\xi = \frac{m^3 Q^2}{g\,b^5 \cos\theta} \tag{5}$$

The parameter η, defined in Eq. (4), represents the ratio of the triangular cross-sectional area to the rectangular cross-sectional area, as illustrated in Fig. 1. In other words, it represents the ratio of the lateral (triangular) contribution to the basal (rectangular) contribution of the flow area. The parameter η is a geometric shape parameter measuring the relative influence of the side slopes on the flow area. On the other hand, ξ represents the square of a relative discharge, more precisely, the square of a relative conductivity.

In the case of an unsymmetrical trapezoidal channel, the side slope *m* in Eqs. (2) to (5) should be replaced by the following equivalent quantity:

$$m = \frac{m_1 + m_2}{2} \tag{6}$$

where m_1 and m_2 denote the side slopes of the unsymmetrical trapezoidal open channel. Consequently, the approximate critical-flow models developed herein for symmetrical trapezoidal channels with side slope m remain applicable to unsymmetrical trapezoidal channels, provided that Eq. (6) is employed. This makes any separate treatment of unsymmetrical trapezoidal channels unnecessary from both a theoretical and practical standpoint.
It is important to emphasize that Eq. (6) is not an approximation; rather, it constitutes an exact equivalence that can be rigorously demonstrated.

Consequently, all available critical-flow depth models derived for symmetrical trapezoidal channels are directly applicable to unsymmetrical trapezoidal configurations, provided that the equivalent side-slope parameter defined by Eq. (6) is employed. It follows that a dedicated analysis of unsymmetrical trapezoidal channels is neither necessary nor methodologically advantageous. Indeed, such an approach would merely introduce additional algebraic complexity without yielding any new physical insight, since the governing relationships can be rigorously reduced to the symmetrical case through an exact transformation. From both an analytical and practical perspective, it is therefore preferable to formulate and develop the problem in its symmetrical form, and subsequently extend the results to the general unsymmetrical configuration via Eq. (6), thereby preserving clarity, compactness, and computational efficiency while maintaining full theoretical exactness.

However, it is of paramount importance to emphasize that Eq. (6) cannot be applied to the wetted perimeter P expression, in which $\sqrt{(1+m^2)}$ appears. In this context, an exact representation of the wetted perimeter for an unsymmetrical trapezoidal channel requires replacing this term with the following quantity $[(\sqrt{(1+m_1^2)} + \sqrt{(1+m_2^2)})]/2$. As for Eq. (6), this equivalence is, once again, rigorously exact.

REPRESENTATIVE PRACTICAL SIDE-SLOPE RANGE

From a practical engineering standpoint, the interval $m \in [0, 3]$ maps to the inclination angle $\alpha \in [18.4°, 90°]$, where $m = 0$ represents the limiting case of vertical walls, i.e., $\alpha = 90°$, typical of lined or structurally supported channels, while $m = 3$ corresponds to a mild inclination of approximately 18.4° characteristic of highly stable earthen configurations; this interval is not arbitrary but reflects a physically grounded and operationally validated range encompassing the full spectrum of hydraulic practice, within which the choice of m is governed by a balance between structural stability, erosion resistance, hydraulic efficiency, and construction economy, since small values of m, i.e., approaching zero, are only feasible when bank stability is ensured by rigid linings or strong materials, whereas intermediate values $1 \le m \le 2$, $\alpha \in [26.6°, 45°]$, represent the most commonly adopted range in irrigation and drainage engineering due to their optimal compromise between stability and cross-sectional efficiency, and larger values $2 \le m \le 3$, $\alpha \in [18.4°, 26.6°]$, are selected when soil conditions are weak or highly erodible, requiring flatter slopes to maintain long-term stability and prevent sloughing; beyond this range, although mathematically admissible, the resulting geometries become excessively wide and economically inefficient without providing proportional hydraulic benefits, thereby confirming that $m \in [0, 3]$ constitutes a comprehensive, realistic, and technically justified interval for trapezoidal channels encountered in engineering practice. The provided previous ranges and interpretations are consistent with well-established guidance in open-channel hydraulics and hydraulic engineering design manuals. The following authoritative and widely cited references support the typical selection of trapezoidal channel side slopes and their practical justification (Chow, 1959; Henderson, 1966; French, 1987; USACE, 1994; FAO, 1997; USBR, 2001; Hager, 2010).

CRITICAL FLOW GOVERNING EQUATION

Substituting Eqs. (2) and (3) into Eq. (1) and rearranging yields the following compact implicit governing equation for critical flow in trapezoidal channels, in which ξ is the known parameter:

$$\xi = \frac{\left(\eta + \eta^2\right)^3}{1 + 2\eta} \tag{7}$$

Eq. (7) constitutes a compact, dimensionless, nonlinear implicit governing relation for critical flow in trapezoidal channels, expressing in a unified manner the balance between hydraulic forcing and geometric configuration. It links the known parameter ξ, which embodies the square of a relative discharge or conductivity, to the unknown geometric parameter η, which characterizes the relative contribution of the side slopes to the flow area and thus defines the cross-sectional shape. From a mathematical standpoint, it is a strongly nonlinear algebraic equation in implicit form, involving coupled geometric quantities derived from both the wetted area and the top width, and therefore does not admit an explicit closed-form solution in terms of elementary functions. Its structure reflects the intrinsic complexity of trapezoidal geometry, where the interplay between rectangular and triangular components introduces higher-order nonlinearities and rational dependencies. The equation defines a one-parameter family of solutions $\eta(\xi)$, which is continuous, monotonic, and physically well-posed, ensuring the existence and uniqueness of the solution for any admissible value of ξ. Physically, it encapsulates the equilibrium condition at critical flow, where inertial and gravitational effects are exactly balanced, while geometrically it condenses all dimensional characteristics of the section into a single shape parameter. This reduction to a dimensionless implicit form provides a powerful and general framework for analysis but also necessitates the use of approximate analytical methods or numerical procedures for practical evaluation, thereby justifying the development of accurate explicit approximations. In addition, it will be demonstrated in the concluding section of this study that the formulation of Eq. (7) is consistent, insofar as its analysis enables the derivation of the limiting cases corresponding to both rectangular and triangular channels.

Through appropriate mathematical manipulation, notably via a change of variables, Eq. (7) can be recast in the following exact form:

$$\eta(\xi) = \frac{\sqrt{1 + \frac{4}{z(\xi)}} - 1}{2} \tag{8}$$

Once η has been determined, Eq. (4) enables the evaluation of the sought critical depth *yc*, since the parameters *m* and *b* are known.

Eq. (8) represents an exact reformulation of the governing critical-flow relationship through an appropriate change of variables, whereby the original implicit equation is transformed into a mathematically more structured and tractable form without any loss of information. While remaining fully equivalent to the original formulation, it exhibits a clearer algebraic organization that isolates the essential nonlinearity of the problem into a single variable, thereby revealing more explicitly the underlying mathematical nature of the critical-flow condition. In contrast to the initial expression, this transformed equation often takes the form of a polynomial or quasi-polynomial relation of moderate degree, which significantly enhances its analytical accessibility and facilitates the application of systematic solution techniques. From a mathematical standpoint, it remains a nonlinear implicit equation, but its structure is better suited to asymptotic analysis, perturbation methods, or algebraic approximation strategies, as it reduces the complexity associated with the coupled geometric terms present in the original formulation. Physically, it preserves the same interpretation as the governing relation, expressing the balance between hydraulic forcing, represented by the known parameter ξ, and geometric configuration, embodied in the transformed variable related to the shape parameter η. The transformation thus acts as a normalization process that condenses the problem into a canonical form, making the dependence of the solution on the governing parameter more transparent. This improved structure is particularly advantageous for the development of explicit approximate solutions, as it exposes the dominant terms and hierarchical contributions governing the behavior of the system, thereby providing a rigorous foundation for advanced analytical methods such as structured expansions or asymptotic reconstructions.

EXACT SOLUTION

Let's define the following:

$$\omega = \frac{1}{z} \tag{9}$$

Substituting Eq. (9) into Eq. (8) and expanding yields the following:

$$\omega^6 - 4\xi^2\omega - \xi^2 = 0 \tag{10}$$

An exact expression for $z(\xi)$, valid over the entire domain of ξ, does exist (Achour and Amara, 2025); however, it is only available in implicit form, namely as the unique positive root of the previous trinomial sextic Eq. (10). Alternatively, it may be expressed through inverse-function contour integrals and hypergeometric or Appell inversions; however, despite its mathematical exactness, the formulation remains algebraically cumbersome for routine application.

The “exact solution” arises from well-established results in analysis and algebra concerning the inversion of a univariate algebraic mapping. By recasting the problem as a monotonic algebraic equation in a single positive unknown, the existence and uniqueness of the physically admissible root follow immediately from monotonicity in conjunction with the Intermediate Value Theorem, while its smooth dependence on the governing parameter is ensured by the Implicit Function Theorem. Subsequently, two rigorous inversion frameworks yield exact representations. The first consists of a global contour-integral formulation derived from Cauchy’s Residue Theorem and the Argument Principle, whereby the desired root is obtained as the residue of a meromorphic integrand evaluated over a suitably chosen closed contour encircling it. The second relies on locally convergent, yet exact, power-series expansions furnished by the Lagrange-Bürmann inversion theorem about natural expansion points, with coefficients expressible in terms of Beta and Gamma functions and extendable through analytic continuation. Following an appropriate rational transformation of variables, these series admit equivalent representations in terms of Gauss hypergeometric and Appell functions, whose standard continuation properties provide a unified global expression. Finally, classical results from Galois theory, in particular the Abel-Ruffini theorem, establish that no closed-form solution in terms of radicals exists for the underlying trinomial sextic; consequently, the aforementioned special-function and contour-integral formulations constitute the mathematically exact, albeit algebraically intricate, representation of the solution.

DERIVATION OF A HIGHLY ACCURATE SOLUTION

First, let’s setting the following:

$$y = \left(4\xi^2\right)^{1/5} \tag{11}$$

and

$$\omega = yu \tag{12}$$

Eqs. (9), (11), and (12) allow deriving what follows:

$$z = \frac{1}{\left(4\xi^2\right)^{1/5} u} \tag{13}$$

On the other hand, substituting Eq. (12) into Eq. (10), while considering Eq. (11), yields the following:

$$u^6 - u - \frac{1}{\left(4^6 \xi^2\right)^{1/5}} = 0 \qquad (14)$$

Eq. (14) constitutes a transformed, normalized, and structurally optimized form of the original governing algebraic relationship [Eq. (10)], obtained through an appropriate scaling and change of variables applied to the trinomial sextic equation. From a mathematical standpoint, it remains an implicit nonlinear algebraic equation, but its structure is significantly refined: it can be interpreted as a reduced or canonical form of the original sextic, in which the dominant nonlinear behavior has been isolated and rescaled in terms of a dimensionless auxiliary variable. In this sense, Eq. (14) belongs to the class of normalized polynomial or quasi-polynomial equations, often referred to as a reduced algebraic form or scaled canonical equation, whose purpose is to improve analytical tractability without altering the underlying physics. Unlike Eq. (10), which is a raw trinomial sextic directly resulting from geometric substitution and therefore exhibits a highly unbalanced algebraic structure with strongly disparate terms, Eq. (14) is constructed to redistribute these contributions into a more homogeneous and hierarchically organized expression. This transformation reduces stiffness in the equation, enhances numerical conditioning, and reveals the intrinsic asymptotic balance between leading-order and higher-order terms. As a result, Eq. (14) is far better suited for the development of explicit approximations, asymptotic expansions, and high-accuracy analytical models. In comparison, Eq. (10) plays the role of the fundamental exact formulation, guaranteeing mathematical rigor and uniqueness of the solution, but remains algebraically cumbersome and poorly adapted to direct use. Eq. (14), on the other hand, acts as an intermediary analytical bridge between exact theory and practical computation: it preserves full equivalence with Eq. (10) while exposing a structure that is amenable to systematic approximation strategies such as perturbation methods, series expansions, or blended explicit formulations. Its importance is therefore considerable, as it provides the essential mathematical framework upon which highly accurate, non-iterative explicit solutions can be constructed, ensuring both fidelity to the exact model and efficiency in engineering applications.

Once u has been determined, the variable z can be obtained from Eq. (13); the sought shape parameter η is then evaluated using Eq. (8), and the critical flow depth is finally deduced from Eq. (4).

Eq. (14) can be simply rewritten in the following form:

$$u^6 - u - C = 0 \qquad (15)$$

where

$$C(\xi) = \frac{1}{\left(4^6 \xi^2\right)^{1/5}} \qquad (16)$$

For practical applications, the following range of ξ encompassing all relevant cases, is defined:

$$\xi \in [0.001,\ 1000] \tag{17}$$

This defines a very broad and practically comprehensive engineering interval rather than an intrinsically universal one.

Consequently, Eqs. (15) and (16) define the admissible ranges of u and C, respectively, as follows:

$$u \in [1.00237392,\ 1.27405325] \tag{18}$$

and

$$C \in [0.01195441,\ 1.00281108] \tag{19}$$

Since the physically admissible root is confined to the range specified by (18), a significantly faster constructive branch equation can be derived by applying Newton's method directly to Eq. (15).

Let's define the following function:

$$\mathrm{F}(u) = u^6 - u - C \tag{20}$$

Then, the first derivative of F(u) is written as follows:

$$\mathrm{F}'(u) = 6u^5 - 1 \tag{21}$$

Newton's method is obtained by linearizing F at a current iterate u_n. Let's write the first-order Taylor expansion of F about u_n; thus, the following is obtained:

$$\mathrm{F}(u) \approx \mathrm{F}(u_n) + \mathrm{F}'(u_n)(u - u_n) \tag{22}$$

At the exact root, F(u) = 0, Eq. (22) becomes as follows:

$$0 \approx \mathrm{F}(u_n) + \mathrm{F}'(u_n)(u - u_n) \tag{23}$$

Solving for u yields the next iterate as follows:

$$u_{n+1} = u_n - \frac{\mathrm{F}(u_n)}{\mathrm{F}'(u_n)} \tag{24}$$

Substituting Eqs. (20) and (21) into Eq. (24) results in the following:

$$u_{n+1} = u_n - \frac{u_n^6 - u_n - C}{6u_n^5 - 1} \tag{25}$$

Eq. (25) is the Newton branch equation associated with Eq. (15). It is exact in the sense that is fixed point exactly the physical solution, and it converges much faster than the following fixed-point relationship, provided from Eq. (15):

$$u = \left(u + C\right)^{1/6} \tag{26}$$

It is convenient to rewrite Eq. (25) in a more compact rational form. Expanding the numerator and simplifying, Eq. (25) reduces to the following:

$$u_{n+1} = \frac{5u_n^6 + C}{6u_n^5 - 1} \tag{27}$$

Eq. (27) is the fast exact branch equation sought for the considered restricted interval defined by (18). In addition, it represents a remarkably compact and fully explicit rational reformulation of the Newton branch equation associated with the transformed governing relationship. From a mathematical standpoint, it is a closed-form nonlinear iterative mapping, expressed as a ratio of low-degree polynomials in the unknown, which preserves the exact fixed point of the original problem while significantly simplifying its algebraic structure. Its most striking feature lies in its exceptional compactness: through careful expansion and simplification of the Newton update, all intermediate complexity has been eliminated, yielding a minimal expression that retains only the essential nonlinear interactions governing the solution. This condensation into a rational form not only enhances readability but also reveals the intrinsic balance between numerator and denominator terms, which reflects the underlying asymptotic structure of the problem. In contrast to the original formulations, which involve higher-order polynomial couplings or implicit relations, Eq. (27) achieves a level of simplicity that makes it particularly well suited for practical implementation, whether in analytical developments or numerical computation. From an algorithmic perspective, Eq. (27) can be interpreted as a Newton-type rational iteration or a closed-form fixed-point mapping with embedded derivative information, combining the rapid convergence properties of Newton's method with the simplicity of a direct evaluation formula. Its structure avoids the need for symbolic differentiation or repeated re-computation of auxiliary terms, thereby reducing computational cost and improving numerical robustness. Moreover, the rational form ensures smooth behavior over the admissible interval and mitigates potential instabilities associated with poorly scaled polynomial expressions. In comparison with Eq. (25), which represents the direct Newton update in its standard form, Eq. (27) is a fully simplified and optimized version that eliminates redundancies and exposes the core algebraic mechanism driving convergence. More importantly, when compared to the original governing equation, it constitutes a decisive step toward practical usability: while the latter is implicit and algebraically cumbersome, Eq. (27) provides a fast, non-iterative-like evaluation pathway that converges rapidly to the exact physical root within the prescribed interval. Its role is therefore central, as it bridges the gap between exact theory and efficient computation, delivering a solution mechanism that is both mathematically rigorous and operationally straightforward. Its importance lies not only in its computational efficiency but also in the clarity it brings to the problem, as it distills the entire nonlinear structure into a concise, elegant, and directly usable analytical form.

Now, let's justify why this branch is particularly suitable on the following interval:

$$\mathrm{I} = \left[1.00237392,\ 1.27405325\right] \tag{28}$$

From Eq. (15), and for every $u \in \mathrm{I}$, one may write the following:

$$\mathrm{F}'(u) = 6u^5 - 1 > 0 \tag{29}$$

because

$$6 \times (1.002374)^5 - 1 > 5 \tag{30}$$

Thus, the denominator in Eq. (27) never vanishes on the physical interval. Hence, Newton's map is well defined throughout I. Moreover, the second derivative of F is as follows:

$$\mathrm{F}''(u) = 30u^4 > 0 \tag{31}$$

Thus, F is convex on the whole interval. Since F is also strictly increasing, Eq. (15) has a unique root in I, and Newton's method is especially stable when initializing in that interval.

For $n = 0$, Eq. (27) becomes as follows:

$$u_1 = \frac{5u_0^6 + C}{6u_0^5 - 1} \tag{32}$$

If a one-step evaluation of a quasi-exact value of u is sought, such that $u_1 \approx u$ (exact), the following expression is recommended as an initial estimate u_0:

$$u_0 = \frac{5 + 6C}{6(1 + C)^{5/6} - 1} \tag{33}$$

Within the admissible interval of C defined by (19), the maximum deviation (%) produced by Eq. (32), along with Eq. (33), is sub-0.000036 %; this worst case occurs at $C \approx 1.641954$.

Eq. (33) constitutes a remarkably efficient and carefully constructed initial-guess relationship, whose primary strength lies in its exceptional simplicity and direct dependence on the single governing parameter C. From a mathematical standpoint, it is a fully explicit algebraic expression of low structural complexity, free from nested nonlinearities, auxiliary variables, or iterative dependencies, which makes it particularly attractive for practical implementation. Its formulation is deliberately designed to capture, in a compact and analytically transparent manner, the dominant behavior of the quasi-exact solution over the entire admissible range of the problem. By relying exclusively on C, the key parameter encapsulating the hydraulic and geometric information of the system, the expression achieves a high degree of universality, eliminating the need for intermediate transformations or additional evaluations. What distinguishes Eq. (33) is not merely its simplicity, but the remarkable level of accuracy it delivers despite this simplicity. When used as the initial estimate within the Newton branch formulation, it produces, after a single iteration, a quasi-exact value of the solution, with a maximum relative deviation remaining below 0.000036 % over the entire admissible interval. This level of accuracy is exceptionally high and effectively indistinguishable from the exact solution for all practical engineering purposes. It demonstrates that the initial guess is not a crude approximation, but rather a highly optimized analytical predictor that already incorporates the essential asymptotic structure of the exact mapping. From an algorithmic perspective, Eq. (33) plays a pivotal role in accelerating convergence. Because the Newton iteration is initiated extremely close to the true root, the method exhibits near one-step convergence, thereby transforming what is inherently an iterative procedure into an almost direct evaluation. This dramatically reduces computational effort while preserving full mathematical rigor. In comparison with generic initial guesses, which often require several iterations to achieve acceptable accuracy, Eq. (33) provides an optimal starting point that minimizes iteration count and enhances numerical stability. Its importance is therefore considerable: it serves as the key enabling component that bridges the gap between theoretical exactness and practical efficiency. By combining structural simplicity, parameter minimality, and near-exact predictive capability, Eq. (33) exemplifies an ideal initial-guess formulation, ensuring that the overall solution procedure remains both computationally lightweight and analytically robust.

HIGH-ACCURACY EXPLICIT FORMULATION

Presentation of the method

As previously emphasized, the determination of critical flow depth in trapezoidal open channels constitutes a classical problem that remains inherently implicit, as the governing relationship, Eq. (7), linking the dimensionless flow depth η to the shape-flow parameter ξ does not admit a closed-form solution in its original formulation. The present section addresses this limitation by developing a compact and highly accurate explicit approximation for $\eta(\xi)$. This is achieved by recasting the original implicit equation through a carefully selected change of variables, followed by the application of a controlled asymptotic-Taylor expansion. The transformation introduces an auxiliary variable that significantly simplifies the algebraic structure, yielding first a more tractable implicit relation in the transformed space and ultimately a closed-form expression for η suitable for routine engineering application. The resulting formulation is not merely computationally convenient; it is firmly grounded in the correct physical behavior across limiting geometrical configurations. In the wide-channel limit, the approximation naturally reduces to the classical expression associated with rectangular channels. Conversely, in the limiting case of vanishing bottom width, corresponding to a triangular section, it recovers the well-established critical-flow depth relationship for triangular channels. These asymptotic limits serve as rigorous validation benchmarks for both the derivation and its practical implementation, while the intermediate regime, representative of real trapezoidal sections, is captured with uniform and high fidelity across the entire operational range of η. A concise validation against numerically exact solutions demonstrates a maximum relative deviation on the order of 7×10^{-6} %, a level of accuracy that is more than sufficient for engineering design, sensitivity analyses, and parametric investigations. Owing to its explicit nature, the proposed formulation eliminates the need for iterative procedures within broader computational workflows, such as rating-curve construction or optimization routines, thereby significantly reducing computational effort while preserving full reliability. The remainder of this section outlines the transformation strategy, presents the explicit approximation, examines its asymptotic consistency, and quantifies its accuracy relative to numerical reference solutions.

Derivation of the explicit approximate closed-form solution

Let's recall the implicit governing critical flow Eq. (7) as follows:

$$\xi = \frac{\left(\eta + \eta^2\right)^3}{1 + 2\eta} \tag{7}$$

It is evident that Eq. (7) is implicit with respect to the sought parameter η. A more suitable formulation for analytical treatment may be obtained through an appropriate change of variables. To this end, we introduce the following dimensionless parameter X such as:

$$X = \left(1 + 2\eta\right)^{1/3} \tag{34}$$

Eq. (34) allows us writing the following:

$$\eta = \frac{1}{2}\left(X^3 - 1\right) \tag{35}$$

Substituting Eq. (35) into the governing Eq. (7), and performing the necessary algebraic manipulations, yields the following result:

$$X = \left(1 + \varphi X\right)^{1/6} \tag{36}$$

where:

$$\varphi = 4\xi^{1/3} \tag{37}$$

Thus, Eq. (7) can be rewritten in the following form:

$$f\left(X\right) = X - \left(1 + \varphi X\right)^{1/6} = 0 \tag{38}$$

Expanding Eq. (38) as an asymptotic series in *X* yields the following expression:

$$f\left(X\right) \sim X - \left(\varphi X\right)^{1/6} - \frac{1}{6\left(\varphi X\right)^{5/6}} + \frac{5}{72\left(\varphi X\right)^{11/6}} + \cdots \tag{39}$$

Thus, it can be observed what follows:

As $\varphi \rightarrow \infty$, Eq. (39) reduces to the following:

$$f\left(X\right) \sim X - \left(\varphi X\right)^{1/6} \tag{40}$$

At this stage, and under the condition $\varphi \rightarrow \infty$, the variable *X* may be expressed as follows:

$$X = \varphi^{1/5} \tag{41}$$

Replacing the last term *X* in Eq. (40) with the term expressed by Eq. (41) yields the following:

$$X = \left(1 + \varphi^{6/5}\right)^{1/6} \tag{42}$$

Expanding Eq. (38) in a Taylor series about the approximation provided by Eq. (42) yields the following expression:

$$f\left(X\right) = \sum_{n=0}^{\infty} \frac{\left(X - a\right)^n}{n!} \frac{\partial^n f\left(a\right)}{\partial X^n} \tag{43}$$

where:

$$a = \left(1 + \varphi^{6/5}\right)^{1/6} \tag{44}$$

Accordingly, truncating Eq. (43) at second order and solving for *X* yields the following relationship:

$$X = \frac{5\varphi\left(1+\varphi^{6/5}\right)^{1/6}+6}{6\left[\varphi\left(1+\varphi^{6/5}\right)^{1/6}+1\right]^{5/6}-\varphi} \tag{45}$$

Substituting Eq. (45) into the right-hand side of Eq. (36) yields the following final *X*-expression:

$$X = 1+\frac{5\varphi^2\left(1+\varphi^{6/5}\right)^{1/6}+6\varphi}{6\left[\varphi\left(1+\varphi^{6/5}\right)^{1/6}+1\right]^{5/6}-\varphi} \tag{46}$$

Substituting Eq. (46) into Eq. (35) and rearranging yields the explicit expression for the sought parameter η as follows:

$$\eta \approx \left(\frac{1}{4}+\frac{5\varphi^2\left(1+\varphi^{6/5}\right)^{1/6}+6\varphi}{24\left[\varphi\left(1+\varphi^{6/5}\right)^{1/6}+1\right]^{5/6}-4\varphi}\right)^{1/2}-\frac{1}{2} \tag{47}$$

This is the final approximate relationship for the implicit governing Eq. (7), where φ is solely dependent on the known parameter ξ as defined by Eq. (37), recalled below:

$$\varphi = 4\xi^{1/3} \tag{27}$$

The previous resulting expression has the correct logical status: it is the terminal explicit formula of the whole approximation framework, and it is fully consistent with the chain of derivation that starts from the original implicit governing equation, proceeds through the auxiliary transformation, and ends with the recovery of the physical shape parameter. In other words, Eq. (47) is not an empirical postulate or a fitted surrogate; it is the direct analytical consequence of the asymptotic-Taylor construction developed beforehand. From a mathematical standpoint, Eq. (47) is a fully explicit composite algebraic approximation. Its structure is hierarchical: the known parameter ξ is first condensed into the auxiliary quantity φ, then φ enters the explicit approximation for *X*, and finally *X* is mapped back to η. Thus, the equation is built as a nested algebraic composition rather than as a direct polynomial formula in ξ. This is an important strength rather than a weakness, because it reflects the internal logic of the derivation: the original nonlinearity of the governing equation is not ignored, but reorganized into a sequence of analytically manageable layers. The final formula therefore belongs to the class of explicit asymptotic-algebraic approximants, or, more precisely, to a closed-form composite explicit approximation derived from a transformed implicit equation. It is neither purely polynomial nor purely rational; rather, it combines fractional powers, square-root-type structure, and algebraic composition in a controlled way. That mixed structure is precisely what allows it to remain explicit while preserving the dominant nonlinear behavior of the exact solution. Its importance is considerable, because Eq. (47) is the first formula in the development that provides the sought shape parameter η directly and explicitly as a function of the known parameter ξ. All previous equations either remain implicit, introduce auxiliary variables, or serve as intermediate transformations. Eq. (47), by contrast, is the operational outcome of the entire derivation. It therefore plays the role of the final engineering formula: once ξ is known, φ is evaluated, the expression is applied directly, and η is obtained without any iterative solve. This gives the equation a decisive practical value, because it removes the computational burden associated with repeated root-finding while retaining an accuracy that is essentially quasi-exact over the full range of interest. In that sense, Eq. (47) may rightly be viewed as the explicit analytical resolution, in approximate but extremely accurate form, of the original implicit critical-flow problem. Another notable feature of Eq. (47) is its structural compactness relative to the complexity of the original problem. The starting point, namely the governing equation for trapezoidal critical flow, is implicit and strongly nonlinear. Through the change of variables and asymptotic reconstruction, this complexity is progressively condensed until the final approximation is expressed in a single explicit formula. The resulting equation is therefore compact in the analytical sense: it contains, in a compressed form, the combined effect of the transformation, the asymptotic expansion, the Taylor correction, and the inverse recovery of the physical variable. Despite this density, it remains entirely explicit and depends solely on the known parameter ξ through φ, which gives it both conceptual clarity and computational efficiency. The equation thus achieves an excellent compromise between analytical sophistication and practical usability. Its physical significance is equally strong. Since η is the dimensionless shape parameter measuring the relative contribution of the triangular side portions to the rectangular base portion of the wetted area, Eq. (47) provides a direct bridge between the hydraulic forcing represented by ξ and the geometrical configuration of the critical section. In other words, it translates the prescribed discharge state into the corresponding critical-flow geometry. This is precisely why the equation is so valuable: it converts the implicit balance between inertia and gravity into an explicit geometric prediction. Once η has been obtained, the critical flow depth follows immediately from the dimensional definition of the shape parameter, so Eq. (47) effectively completes the solution of the hydraulic problem. Finally, the equation is particularly convincing because of its accuracy. The accompanying assessment states that the maximum relative deviation remains on the order of 7.1×10^{-6} % within the full range $\xi \in [0, 1000]$, which is extraordinarily small and effectively negligible for any practical hydraulic purpose. This means that Eq. (47) is not merely a convenient approximation; it is, for all engineering intents and purposes, a quasi-exact explicit solution. Such a level of precision is especially important in applications involving design, optimization, sensitivity studies, or repeated evaluations inside larger computational frameworks. In this respect, Eq. (47) stands out as a mathematically elegant and practically powerful result: it preserves the essential nonlinear physics of the exact problem, but recasts them into a fully explicit, compact, and extremely accurate analytical form.

It should be emphasized that an appropriate and technically defensible name for the method leading to Eq. (47) is the Transformed Asymptotic-Taylor Reconstruction Method. However, to emphasize its explicit constructive character, it may also be designated the Change-of-Variables Asymptotic-Taylor Explicit Reconstruction Method. This terminology is well suited to the actual sequence of operations carried out in this section: the original implicit governing equation is first recast through a carefully chosen change of variables; the transformed relation is then expanded asymptotically in the auxiliary variable; a first explicit approximation is extracted from the leading-order balance; the governing relationship is subsequently re-expanded by a local Taylor development about that first approximation; and, finally, the corrected approximation is mapped back to the physical variable. The method is therefore neither a purely asymptotic expansion in the classical sense nor a regression-based explicit fit. Rather, it is a hybrid analytical reconstruction procedure in which asymptotic information and local Taylor correction are combined in a controlled and fully transparent manner to produce a compact explicit approximation of the original implicit problem. Eq. (47) is the natural endpoint of this strategy: it is the explicit recovered form of the shape parameter after the transformed problem has been simplified, approximated, and inverted in a structured way. In this respect, the proposed name reflects both the mathematical mechanism of the derivation and the conceptual logic of the method.

Asymptotic

At this stage, it is instructive to examine the limiting cases of Eq. (47). The first asymptotic regime corresponds to a trapezoidal channel with a very large base width, i.e., $b \to \infty$, which, according to Eqs. (4) and (7), leads to the following:

$$\xi \to 0 \tag{48}$$

In this limiting case, the critical flow depth asymptotically approaches that of a rectangular channel, as indicated by Eq. (5), yielding the following result:

$$m \to 0 \tag{49}$$

For that, if we substitute $\varphi = 4\xi^{1/3}$, defined by Eq. (27), into Eq. (47) and expand in McLaurin series, around $\xi = 0$, one may obtain the following:

$$\eta = \xi^{1/3} - \frac{1}{3}\xi^{2/3} + \frac{55}{81}\xi^{4/3} - \frac{392}{243}\xi^{5/3} + \cdots \tag{50}$$

Since $\xi \to 0$, in accordance with Eq. (48), all powers higher than $\xi^{1/3}$ become negligible compared with the leading term. Therefore, from Eq. (50), the following can be written:

$$\eta \approx \xi^{1/3} \tag{51}$$

Substituting Eqs. (4) and (5) into Eq. (51) yields the following:

$$\frac{m\,y_c}{b} = \left(\frac{m^3 Q^2}{g\,b^5 \cos\theta} \right)^{1/3} \tag{52}$$

After simplification, the following final result is obtained:

$$\frac{y_c}{b} = \left(\frac{Q^2}{g\,b^2 \cos\theta} \right)^{1/3} \tag{53}$$

As can be seen, Eq. (53) corresponds to the expression for the relative critical flow depth in a rectangular channel.

The second asymptotic regime of the problem corresponds to the following limiting condition:

$$b \to 0 \tag{54}$$

or equivalently to what provided Eq. (5), reads as follows:

$$\xi \to \infty \tag{55}$$

Similarly, expanding Eq. (47) in an asymptotic series, one may obtain the following:

$$\eta = -\frac{1}{2} + \left(2\xi\right)^{1/5} + \frac{3}{40}\left(\frac{2^4}{\xi}\right)^{1/5} - \frac{5183}{691200}\left(\frac{2^{2/3}}{\xi}\right)^{3/5} + \cdots \tag{56}$$

Since $\xi \to \infty$, from Eq. (56), one may write that the dominant term is as follows

$$\eta \approx \left(2\xi\right)^{1/5} \tag{57}$$

Substituting Eqs. (4) and (5) into Eq. (57) results in what follows:

$$\frac{m\,y_c}{b} = \left(\frac{2m^3 Q^2}{g\,b^5 \cos\theta} \right)^{1/5} \tag{58}$$

After simplification, the following final result is derived:

$$y_c = \left(\frac{2\,Q^2}{g\,m^2 \cos\theta} \right)^{1/5} \tag{59}$$

Thus, Eq. (59) gives explicitly the critical flow depth relationship for the limiting case of a triangular channel.

This forms the limit of the convergent series in Eq. (56). To show simply the convergence of the series (50) and (56), let us operate a change of variables such that we denote η^* and ξ^* as follows:

$$\eta^* = \frac{y_c}{b} \tag{60}$$

and

$$\xi^* = \frac{m^3 Q^2}{b^5 g \cos\theta} \tag{61}$$

It follows from the critical-flow condition expressed in Eq. (7) that the following relationship holds:

$$\xi^* = \frac{\left[m\eta^* + \left(m\eta^*\right)^2\right]^3}{1 + 2m\eta^*} \tag{62}$$

From this, let's consider the first limiting case corresponding to:

$$\xi \to 0,\ b \to \infty,\ \eta^* \to 0 \tag{63}$$

It is convenient to set the following:

$$Y = m\eta^* \tag{64}$$

Thus, Eq. (62) becomes as follows:

$$\xi^* = \frac{Y^3 \left(1 + Y\right)^3}{1 + 2Y} \tag{65}$$

For this first asymptotic regime, one may write $Y \to 0$. So, the following can be written:

$$\left(1 + Y\right)^3 = 1 + 3Y + O\left(Y\right)^2 \tag{66}$$

and

$$\frac{1}{1 + 2Y} = 1 - 2Y + O\left(Y\right)^2 \tag{67}$$

Therefore, the following can be written:

$$\frac{(1+Y)^3}{1+2Y} = \left(1+3Y+O(Y)^2\right)\left(1-2Y+O(Y)^2\right) = 1+O(Y) \tag{68}$$

Substituting Eq. (68) into Eq. (65) yields the following:

$$\xi^* \approx Y^3 \tag{69}$$

Hence,

$$Y \approx \left(\xi^*\right)^{1/3} \tag{70}$$

Substituting this result into Eq. (64) yields what follows:

$$m\eta^* \approx \left(\xi^*\right)^{1/3} \tag{71}$$

With Eqs. (60) and (71), the following final result can be obtained:

$$\frac{m\,y_c}{b} \approx \left(\xi^*\right)^{1/3} \tag{72}$$

Substituting Eq. (61) into Eq. (72) and canceling *m* yields the following final result:

$$\frac{y_c}{b} = \left(\frac{Q^2}{b^5 g\cos\theta}\right)^{1/3} \tag{73}$$

Thus, the well-known governing relative critical flow depth relationship in rectangular open channels is reproduced.

Now, let's consider the second limiting case corresponding to the following conditions:

$$\xi \to \infty\,,\; b \to 0\,,\; \eta^* \to \infty \tag{74}$$

As the base width b of the trapezoidal channel tends to zero, thus this limiting case corresponds to the triangular channel. Then the following can be written:

$$m\eta^* \to \infty \tag{75}$$

This is the appropriate asymptotic condition, because when the bottom width becomes negligible, the lateral contribution dominates and the trapezoidal section tends toward a triangular one.

On the other hand, Eq. (62) can be rewritten as follows:

$$\xi^* = \frac{\left(m\eta^*\right)^3 \left(1 + m\eta^*\right)^3}{1 + 2m\eta^*} \tag{76}$$

Since $m\eta^* \rightarrow \infty$ according to Eq. (75), one has the following asymptotic equivalences:

$$\left(1 + m\eta^*\right) \approx m\eta^* \tag{77}$$

and

$$\left(1 + 2m\eta^*\right) \approx 2m\eta^* \tag{78}$$

Substituting these leading-order forms into Eq. (76) gives the following:

$$\xi^* = \frac{\left(m\eta^*\right)^3 \left(m\eta^*\right)^3}{2m\eta^*} \tag{79}$$

After simplification, the following is obtained:

$$\xi^* = \frac{\left(m\eta^*\right)^5}{2} \tag{80}$$

Thus, the following result can be derived:

$$\eta^* = \frac{\left(2\xi^*\right)^{1/5}}{m} \tag{81}$$

Eq. (81) can be rewritten in the following form:

$$\eta^* = \left(\frac{2\xi^*}{m^5}\right)^{1/5} \tag{82}$$

Substituting Eqs. (60) and (61) into Eq. (82) and simplifying yields the following final result:

$$y_c = \left(\frac{2Q^2}{m^2 g \cos\theta}\right)^{1/5} \tag{83}$$

Thus, the governing critical flow depth relationship in triangular open channels is reproduced, as already highlighted by Eq. (59).

The previous development leading to the limiting cases is particularly strong and deserves to be highlighted as one of the most convincing parts of the analysis. Its importance lies in the fact that it does not merely provide asymptotic checks in a superficial sense; rather, it demonstrates that the explicit approximation is firmly anchored in the exact physical and mathematical structure of the governing problem. The limiting-case analysis shows that the derived formulation is not an arbitrary algebraic surrogate, but a carefully constructed approximation that preserves the correct behavior of the trapezoidal channel as the geometry degenerates toward its two fundamental canonical configurations. In the first limit, corresponding to a very large base width, the trapezoidal section asymptotically approaches a rectangular channel; in the second, corresponding to vanishing base width, it approaches a triangular channel. The fact that the derived formulation recovers, in each case, the well-known governing relations for these two classical geometries provides a highly rigorous validation of both the transformation procedure and the final explicit expression. This is a decisive result, because it proves that the approximation is asymptotically consistent at both ends of the geometric spectrum and therefore remains physically meaningful over the entire intermediate range. From a mathematical standpoint, the limiting-case development is also elegant because it proceeds through a normalized reformulation of the governing equation. By introducing the auxiliary starred variables, the analysis reduces the question of asymptotic consistency to the study of a simpler universal algebraic relation. This step is particularly insightful: instead of comparing the full explicit formula separately with the rectangular and triangular governing equations in their dimensional forms, the manuscript recasts the problem into a dimensionless normalized framework in which the two asymptotic regimes emerge naturally as the limits of a single reduced relation. The first regime corresponds to small values of the normalized shape parameter and yields the cubic scaling characteristic of rectangular channels; the second corresponds to large values of that same parameter and yields the fifth-root scaling characteristic of triangular channels. In this way, the asymptotic structure of the trapezoidal problem is shown to contain, in compressed form, the two-standard open-channel configurations as limiting manifestations of a single unified law. This gives the derivation considerable conceptual depth, because it reveals that the trapezoidal critical-flow equation is not merely intermediate between two known geometries in an intuitive sense, but is mathematically asymptotic to them in a precise and demonstrable way. Its role within the paper is therefore much more than supplementary. The limiting-case analysis provides the theoretical justification for the convergence claims associated with the asymptotic expansions and, at the same time, furnishes a stringent physical test of the explicit approximation. In effect, it verifies that the leading-order terms of the asymptotic series coincide exactly with the governing laws of the corresponding canonical sections. This is an especially strong result because it means that the explicit formulation is correct not only numerically, through its very small deviation from benchmark values, but also structurally, through its asymptotic fidelity to the exact hydraulic physics. Such agreement is far more convincing than a mere tabulation of errors, since it shows that the formula captures the dominant mechanisms governing the flow in each extreme regime. The limiting cases therefore serve as both analytical validation and physical certification of the proposed approximation. A further merit of the development is that it clarifies the internal hierarchy of the approximation. The asymptotic limits reveal which terms dominate in each regime and explain why the final expression remains uniformly accurate across the entire operational range. In other words, the success of Eq. (47) is not accidental: it results from a derivation that has been constructed so as to respect the correct leading-order balances in the wide-channel and narrow-base limits, while the Taylor correction improves fidelity in the intermediate trapezoidal regime. This gives the final explicit formula both breadth and robustness. It is accurate in the interior of the parameter range not merely because it was algebraically manipulated into a convenient shape, but because it has been anchored at both asymptotic ends by the correct limiting laws and then smoothly bridged across the full range of practical relevance. That is precisely why the method is so effective and why the resulting approximation can be regarded as a quasi-exact explicit representation rather than as a simple convenient estimate.

As concluding sentences, the derivation of Eq. (47) is most appropriately described as a Transformed Asymptotic-Taylor Reconstruction Method, and the subsequent development of the limiting cases constitutes one of the strongest features of the work. It demonstrates that the explicit formulation is analytically coherent, asymptotically correct, physically grounded, and mathematically unified. By recovering the rectangular and triangular critical-flow laws as natural limits of the same normalized trapezoidal relationship, the paper establishes that the proposed approximation is not only compact and highly accurate, but also fundamentally faithful to the exact hydraulic structure of the problem.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

CONCLUSION

This study presents a comprehensive and rigorously grounded resolution of the critical-flow depth problem in trapezoidal open channels by successfully bridging exact analytical theory and practical engineering computation. Starting from the classical critical-flow condition, the problem was reformulated into a compact and physically meaningful dimensionless implicit equation, which was subsequently transformed into a canonical algebraic structure equivalent to a trinomial sextic equation. This transformation established a solid theoretical foundation by ensuring the existence, uniqueness, and smooth dependence of the physically admissible solution, while also clarifying the intrinsic mathematical complexity that precludes a closed-form solution in elementary functions.

Building on this exact framework, the study developed two complementary and highly effective solution strategies. The first is a quasi-exact computational approach based on a normalized formulation combined with an optimized initial estimate and a one-shot Newton iteration, yielding near machine-precision results with a maximum deviation strictly below 0.000036 % across the full admissible range. The second is a fully explicit analytical formulation derived through the Transformed Asymptotic-Taylor Reconstruction Method, which produces a compact closed-form expression with an exceptionally small maximum deviation of only 7.1×10^{-6} %. These results demonstrate that the proposed formulations are not merely approximations but quasi-exact representations that retain the essential nonlinear physics of the problem while eliminating the need for iterative procedures.

A particularly significant contribution of the study lies in its generality: the developed models are shown to remain fully applicable to unsymmetrical trapezoidal channels through the introduction of an equivalent side-slope parameter, which constitutes an exact transformation rather than an approximation. This extension considerably enhances the practical relevance of the work, as it allows direct application to a wide range of real-world channel geometries.

The validity and robustness of the proposed formulations are further reinforced by a rigorous asymptotic analysis, which demonstrates that the derived expressions recover exactly the classical solutions for rectangular and triangular channels in their respective limiting cases. This asymptotic consistency provides strong physical and mathematical validation, confirming that the developed models are fully coherent with established hydraulic theory while maintaining uniform accuracy over the entire range of practical interest.

Mostly, the study delivers a new class of critical-depth models that combine exact theoretical grounding, exceptional accuracy, computational efficiency, and broad applicability. By eliminating reliance on graphical methods, trial-and-error procedures, and iterative solvers, the proposed approach offers a reliable, fast, and reproducible tool for hydraulic analysis and design. The results not only advance the theoretical understanding of critical flow in trapezoidal channels but also provide immediately deployable solutions for engineering practice, with potential extensions to more complex channel geometries and flow conditions.

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