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EFFICIENT COMPUTATION OF NORMAL FLOW DEPTH IN CIRCULAR CONDUITS WITHIN THE DARCY-WEISBACH-RMM FRAMEWORK

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ABSTRACT

The determination of normal flow depth in circular conduits remains a fundamental problem in hydraulic engineering, yet most existing methods rely on resistance equations, namely Manning, Chezy, or Darcy-Weisbach, that introduce conceptual inconsistencies when used in dimensionless formulations. These classical approaches require resistance coefficients that, in practice, depend on the unknown normal flow depth itself, thereby compromising both physical coherence and computational reliability. To overcome this limitation, the present study adopts the Darcy-Weisbach relationship within the rigorous framework of the Rough Model Method (RMM), which entirely eliminates the circular dependence on empirical resistance parameters. By transforming the governing hydraulic equation into a dimensionally consistent implicit relationship, the analysis yields a robust expression for the relative normal flow depth ξ_R in the rough reference circular conduit model.

The mathematical arrangement of the final expression ensures numerical stability across the full admissible range of the relative conductivity $Q^* \in [0, \pi]$, This makes it ideally compatible not only with conventional fixed-point approaches but also with convergence-enhancing accelerators, including Aitken's Δ^2 process, yielding highly accurate results with minimal computational effort."

Owing to the severe nonlinearity of the relationship, classical analytical tools, such as the Lagrange-Burmann inversion theorem, prove ineffective because the resulting series converges only for extremely small values of the relative conductivity Q^*. To circumvent these intrinsic limitations, the present work provides a comprehensive numerical reference Table containing exact solutions of ξ_R for $Q^* \in [0, \pi]$, computed with high precision using a reliable bracketing approach. This tabulation, combined with simple linear interpolation, enables practitioners to determine the sought normal flow depth directly and with high accuracy, without iterative computation.

The proposed methodology is conceptually rigorous, computationally efficient, and entirely free from empirical assumptions. The resulting tool is therefore well suited for engineering design, performance evaluation, and operational hydraulic analysis involving circular conduits under uniform flow conditions.

Keywords: Normal flow depth; Circular conduits; Darcy-Weisbach relationship; Rough Model Method (RMM); Linear interpolation, Convergence analysis.

INTRODUCTION

The determination of normal flow depth has long represented a central and recurrent challenge in hydraulic engineering. Classical contributions to this field were primarily graphical, such as those presented by Chow (1973), French (1986), and Sinniger and Hager (1989), while later developments transitioned toward iterative numerical schemes or approximate explicit expressions intended to simplify computation. Despite these efforts, the efficient and physically consistent evaluation of normal flow depth continues to be a topic of considerable interest.

Much of the available literature, particularly the widely cited works of Swamee (1994), Swamee and Rathie (2004), and more recently Amara and Achour (2023), has attempted to derive practical implicit formulas for the normal flow depth in some open channels' profiles based on the Manning or Chezy uniform-flow equations. Although convenient in appearance, these approaches are built on a fundamental conceptual contradiction: both Manning's roughness coefficient *n* and Chezy's coefficient *C* are routinely assumed to be constant, whereas substantial theoretical and experimental evidence has shown that these parameters depend directly on the yet-unknown normal flow depth. The investigations of Achour and Bedjaoui (2006) and Achour and Amara (2020a; 2020b) made this dependency explicit, demonstrating that the resistance coefficient varies systematically with hydraulic radius, flow regime, and boundary roughness.

Consequently, any dimensionless formulation derived from the Manning or Chezy equations implicitly fixes a resistance coefficient that is, in reality, a function of the very quantity being solved for. This circular dependence undermines the physical validity of the resulting formulas and casts doubt on their reliability, regardless of whether they are expressed in implicit or explicit form.

To overcome this inherent inconsistency, a decisive methodological shift was introduced through the adoption of the Darcy−Weisbach formulation integrated with the Rough Model Method (RMM) developed by Achour and colleagues (Achour and Bedjaoui, 2006; Achour, 2007; Lakehal and Achour, 2014; Achour and Sehtal, 2014; Sehtal and Achour, 2023). The RMM replaces unstable empirical roughness parameters with a rigorously defined rough reference model, for which the friction factor becomes a constant under fully rough turbulent conditions. This approach removes the circular dependency on depth-dependent resistance coefficients and yields a fully self-contained, dimensionally consistent implicit relationship for the relative normal flow depth.

Within this modern and physically coherent framework, the normal-flow depth problem can be addressed without recourse to empirical adjustments, iterative friction-factor evaluation, or assumptions about the behaviour of Manning or Chezy coefficients. Instead, the hydrodynamic parameters required by the RMM, geometric dimensions, discharge, gravitational acceleration, slope, kinematic viscosity, and absolute roughness, are all directly measurable and independent of the unknown depth.

Building on this conceptual foundation, the present study focuses on the computation of normal flow depth in circular conduits. The analysis employs the Darcy-Weisbach-RMM formulation to derive a robust implicit equation for the reduced normal depth. Particular attention is devoted to its mathematical structure, which, unlike classical formulations, is inherently suited for stable numerical evaluation across the entire admissible domain. This facilitates the development of reliable iterative procedures and, when necessary, the construction of high-accuracy tabulated solutions for engineering practice.

The governing relationship has been derived and arranged in a form that is exceptionally well adapted to numerical resolution. Its final structure is algebraically simple, dimensionally consistent, and fully self-contained, making it ideal for implementation through iterative schemes. In particular, the equation is naturally compatible with the fixed-point method, which can be applied directly without requiring supplementary transformations or restrictive assumptions. Moreover, the formulation readily accommodates convergence-enhancement strategies such as Aitken's Δ^2 acceleration, ensuring rapid and stable convergence even in regions where the nonlinearity of the hydraulic relationship is pronounced. This deliberate structuring not only guarantees numerical robustness across the full admissible range of relative conductivity but also enables practitioners to obtain highly accurate solutions with minimal computational effort.

However, to provide a tool that is both robust and accessible to practitioners, the present study adopts a fundamentally different approach based on high-precision numerical tabulation combined with simple linear interpolation. By generating a dense reference Table of exact solutions, computed with a highly stable bracketing technique, the method guarantees that each admissible value of the relative conductivity Q^* immediately corresponds to a unique, physically consistent value of the relative normal flow depth ξ_R in the rough reference model. Linear interpolation between tabulated values then provides an extremely accurate approximation, with deviations typically far below engineering-tolerance thresholds.

This approach offers key advantages: it is universally convergent, entirely avoids sensitivity to initial guesses, and eliminates the algebraic and computational complexity associated with solving the implicit equation directly. In addition, because the tabulated function $z(Q^*)$ is smooth and almost linear over short intervals, a simple first-order interpolation is sufficient to reproduce the exact depth with negligible error. As a result, the interpolation-based procedure combines mathematical reliability with operational simplicity, making it particularly suitable for hydraulic design offices, engineering practitioners, and software implementations requiring fast and stable evaluation of the normal flow depth.

Statement of the Problem

The determination of the normal flow depth remains one of the essential tasks in hydraulic engineering, as it governs the conveyance capacity, energy distribution, and overall hydraulic performance of channels and conduits operating under uniform-flow conditions. In the specific case of circular conduits, where the geometry is defined solely by the conduit diameter D (Fig. 1), the normal flow depth directly controls both hydraulic efficiency and flow characteristics.

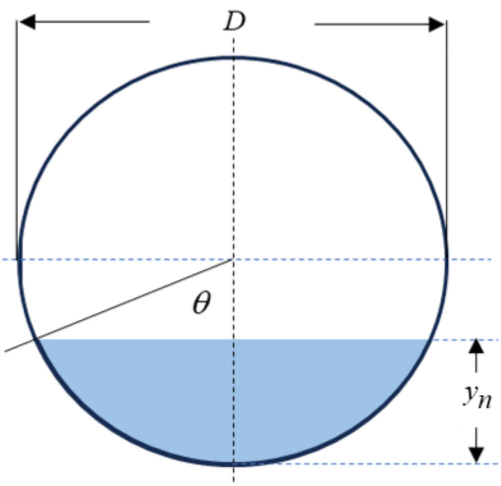

Figure 1: Definition sketch of normal flow depth in a circular conduit

The flow is driven by a set of measurable quantities: the discharge Q, the conduit diameter D, the longitudinal slope S_0, and the fluid kinematic viscosity ν. In addition, the absolute roughness ε, which characterises the height and distribution of surface irregularities along the conduit wall, plays a key role in determining frictional resistance.

Traditional methods for computing normal flow depth rely heavily on classical resistance equations such as those of Darcy−Weisbach, Chezy, or Manning. However, each of these approaches presents significant conceptual and practical limitations. The Darcy−Weisbach relationship, though theoretically robust, requires the evaluation of the Colebrook−White friction factor, which is implicit and necessitates iterative resolution or approximate graphical interpretation. The Chezy and Manning formulas are often preferred because of their apparent simplicity, yet both depend on resistance coefficients, C for Chezy and n for Manning, that are frequently assumed constant in engineering practice.

Such an assumption is fundamentally incorrect. Numerous experimental and theoretical investigations, including the works of Achour and Amara (2020a; 2020b), have demonstrated that these resistance coefficients depend not only on the conduit roughness and fluid properties, but also directly on the unknown normal flow depth itself. This flow depth-dependence makes it physically inconsistent to prescribe fixed values of C or n when calculating the flow depth, thereby introducing a circular contradiction into the computation process.

To overcome this difficulty, the present study adopts the Rough Model Method (RMM), which provides a physically consistent basis for analysing turbulent flow in rough-walled conduits. The RMM replaces empirically uncertain resistance coefficients with a hydraulically equivalent rough reference model, within which the friction factor becomes a constant under fully rough turbulent conditions. This eliminates the need for empirical adjustments, ensures dimensional coherence, and allows the normal flow depth to be computed solely from parameters that are measurable with confidence: the discharge, geometric characteristics, slope, fluid viscosity, and absolute roughness.

Within this framework, the original problem is reformulated into the determination of the relative normal flow depth $\xi = yn/D$, which is extracted from a dimensionless implicit equation derived from the Darcy−Weisbach relationship under the RMM assumptions. This formulation avoids all inconsistencies inherent in classical methods, restores physical correctness to the analysis, and forms the foundation for the computational methodology developed in the remainder of the study.

RMM basic equations

The present analysis is grounded in the classical principles of uniform-flow hydraulics and draws upon three fundamental relationships: the Darcy−Weisbach equation (Darcy, 1854), the Colebrook−White formulation for turbulent resistance (Colebrook, 1939), and the definition of the Reynolds number. In this framework, the energy slope S_f, where the subscript 'f' denotes friction, is taken to be identical to the geometric conduit slope S_0. This equivalence arises directly from the uniform-flow condition, under which the gravitational driving force acting along the conduit axis is precisely counterbalanced by frictional losses along the wetted perimeter. Consequently, the energy gradient may, without loss of generality, be replaced by the conduit slope S_0 throughout the derivation. On this basis, the slope S_0 of an open channel or closed conduit is governed by the following well-established Darcy−Weisbach formulation (Darcy, 1854). Although this relationship was originally developed for fully pressurized circular conduits, Sinniger and Hager (1989) emphasize that its applicability extends to open-channel flows as well, regardless of geometric configuration.

$$S_0 = \frac{f}{D_h} \frac{Q^2}{2 g A^2} \tag{1}$$

where Q denotes the discharge, g the gravitational acceleration, A the wetted flow area, D_h the hydraulic diameter, and f the friction factor. The latter is evaluated using the well-known Colebrook-White relationship (Colebrook, 1939) as follows:

$$f = -2\log_{10}\left(\frac{\varepsilon/D_h}{3.7} + \frac{2.51}{R_e\sqrt{f}}\right) \tag{2}$$

in which ε represents the absolute roughness and Re the Reynolds number. The Reynolds number is in turn expressed as follows:

$$R_e = \frac{4Q}{P\nu} \tag{3}$$

where ν is the kinematic viscosity and P the wetted perimeter.

These three foundational relationships constitute the analytical cornerstone of the Rough Model Method (RMM) and provide the framework from which the dimensionless implicit equation governing the normal flow depth is derived. Together, they enable the formulation of a physically consistent and self-contained expression for the reduced normal flow depth, independent of empirical roughness parameters.

Geometric and hydraulic characteristics of the rough reference model

In the RMM, all geometric and hydraulic properties of the rough reference model are identified by a subscript "R" that simply denotes "Rough". The model is defined by prescribing an arbitrarily chosen relative roughness of

$$\frac{\varepsilon_R}{D_{h,R}} = 0.037 \tag{4}$$

This relatively large relative roughness ensures that the flow regime within the rough model is really "fully rough". Under such conditions, and according to the Colebrook-White formulation, the friction factor attains the following constant value

$$f_R = \frac{1}{16} \tag{5}$$

corresponding to the Reynolds number $R_e = R_{e,R} \rightarrow \infty$.

The rough reference model is further characterized by the following characteristics:

(1) the diameter D_R such as $D = D_R$

(2) a longitudinal slope $S_{0,R} = S_0$

(3) the discharge $Q_R = Q$, implying the following

(4) $y_{n,R} \neq y_n$, and even

(5) $y_{n,R} > y_n$

(6) the aspect ratio, also known as the dimensionless normal flow depth is thus:

$$\xi_R = \frac{y_{n,R}}{D} \neq \xi = \frac{y_n}{D}, \; 0 \leq \xi_R \leq 1$$

(7) the hydraulic diameter $D_{h,R} = \dfrac{4A_R}{P_R}$

(8) The wetted perimeter is as follows:

$$P_R = D\cos^{-1}\left(1 - 2\xi_R\right)$$

(9) The wetted flow area is expressed as follows:

$$A_R = \frac{D^2}{4}\left[\cos^{-1}\left(1 - 2\xi_R\right) - 2\left(1 - 2\xi_R\right)\sqrt{\xi_R\left(1 - \xi_R\right)}\right]$$

(10) The Reynolds number is expressed as follows:

$$R_R = \frac{4Q}{P_R\,\nu}$$

RMM normal flow depth governing relationship

Applying Eq. (1) to the rough reference model yields the following:

$$S_0 = \frac{f_R}{D_{h,R}} \frac{Q^2}{2gA_R^2} \tag{6}$$

Considering the overmentioned characteristics of the rough reference model, and rearranging, Eq. (6) reduces to what follows:

$$\frac{\cos^{-1}\left(1 - 2\xi_R\right)}{\left[\cos^{-1}\left(1 - 2\xi_R\right) - 2\left(1 - 2\xi_R\right)\sqrt{\xi_R\left(1 - \xi_R\right)}\right]^3}\left(\frac{Q}{\sqrt{2gS_0D^5}}\right)^2 = 1 \tag{7}$$

Eq. (7) can be rewritten in the following form:

$$\frac{\sqrt{\cos^{-1}\left(1-2\xi_R\right)}}{\left[\cos^{-1}\left(1-2\xi_R\right)-2\left(1-2\xi_R\right)\sqrt{\xi_R\left(1-\xi_R\right)}\right]^{3/2}}Q^{*}=1 \tag{8}$$

Where Q^* is the relative conductivity expressed as follows:

$$Q^{*}=\frac{Q}{\sqrt{2gS_0D^5}} \tag{9}$$

To simplify calculation, let's define the following:

$$M=\cos^{-1}\left(1-2\xi_R\right) \tag{10}$$

$$N=2\left(1-2\xi_R\right)\sqrt{\xi_R\left(1-\xi_R\right)} \tag{11}$$

Thus, one may rewrite Eq. (8) in the following compact form:

$$\frac{\sqrt{M}}{\left(M-N\right)^{3/2}}Q^{*}=1 \tag{12}$$

Eq. (12) allows writing what follows:

$$\left(Q^{*}\right)^{2/3}=\frac{\left(M-N\right)}{M^{1/3}} \tag{13}$$

Hence:

$$M=\left(Q^{*}\right)^{2/3}M^{1/3}+N \tag{14}$$

Substituting Eqs. (10) and (11) into Eq. (14) yields the following:

$$\cos^{-1}\left(1-2\xi_R\right)=\left(Q^{*}\right)^{2/3}\left[\cos^{-1}\left(1-2\xi_R\right)\right]^{1/3}+ 2\left(1-2\xi_R\right)\sqrt{\xi_R\left(1-\xi_R\right)} \tag{15}$$

Let's take the cosine of both sides; thus, the following can be written:

$$1 - 2\xi_R = \cos\left\{\left(Q^*\right)^{2/3}\left[\cos^{-1}\left(1 - 2\xi_R\right)\right]^{1/3} + 2\left(1 - 2\xi_R\right)\sqrt{\xi_R\left(1 - \xi_R\right)}\right\} \quad (16)$$

Thus, isolating ξ_R, the following final form is derived:

$$\xi_R = \frac{1}{2} - \frac{1}{2}\cos\left\{\left(Q^*\right)^{2/3}\left[\cos^{-1}\left(1 - 2\xi_R\right)\right]^{1/3} + 2\left(1 - 2\xi_R\right)\sqrt{\xi_R\left(1 - \xi_R\right)}\right\} \quad (17)$$

Eq. (17) is the implicit RMM governing equation for ξ_R. From a mathematical standpoint, this equation is best classified as an implicit nonlinear algebraic fixed-point equation. It allows computing the maximum relative conductivity at the upper bound, i.e., $\xi_R = 1$. The calculation shows that, for this end-point value, the maximum admissible relative conductivity is as follows:

$$Q^*_{max} = \pi \quad (18)$$

However, it is worth noting that the plotted curve of Q^* as a function of the relative normal depth ξ_R (Fig. 2) reveals a non-monotonic relationship, which is the key to understanding why two distinct values of ξ_R correspond to the same relative conductivity $Q^* = \pi$.

The curve initially rises steeply from the origin, increases smoothly across most of the relative flow depth range, reaches a distinct maximum slightly before full flow depth, and then bends downward as ξ_R approaches 1. This characteristic shape is fully consistent with the well-known hydraulic behaviour of circular conduits operating under free-surface conditions.

The important consequence visible in the figure is that the vertical line $Q^* = \pi$ intersects the curve at two different points: one at $\xi_R \approx 0.85245$ (*a* on the curve), on the increasing branch, and one at $\xi_R = 1$ (*c* on the curve), on the decreasing branch.

This means that the discharge $Q^* = \pi$ can physically occur at two different normalized depths: one corresponding to a partially filled conduit, the other corresponding to a completely full conduit. The existence of these two intersection points reflects the classical hydraulic fact that a circular pipe conveys its maximum discharge at a flow depth slightly less than full, typically between 93% and 98% of full flow depth. Beyond this point, increasing the flow depth reduces the hydraulic radius and causes the discharge capacity to decrease again, even though the cross-sectional area increases.

Thus, the figure graphically confirms the analytical conclusion: the governing implicit Eq. (17) admits two distinct solutions for ξ_R when $Q^* = \pi$.

This also explains why some iterative numerical methods can converge to a *wrong branch* unless the iteration scheme is carefully designed or the initial guess is appropriately constrained. The figure therefore not only demonstrates the non-monotonic behaviour of the circular-conduit relative conductivity function but also highlights the necessity of using robust root-selection strategies when solving the implicit equation.

Note that, in the present study, the part *abc* of the curve in Fig. 2 is excluded from the analysis.

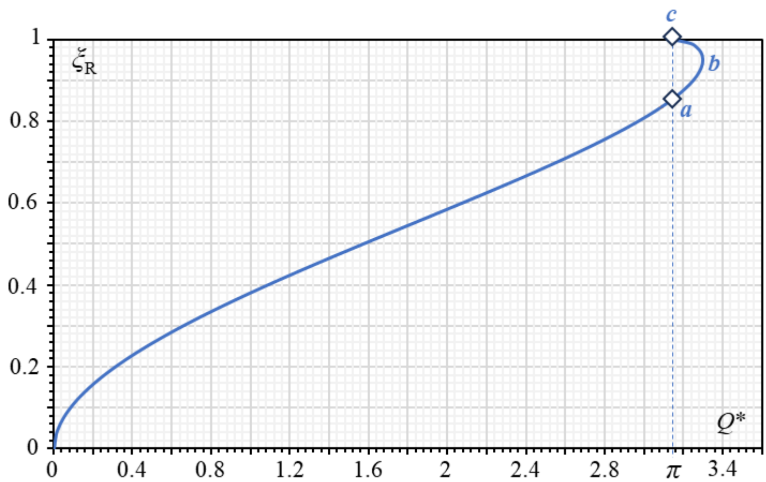

Figure 2: Variation in the relative normal flow depth ξ_R with respect to the relative conductivity according to the governing Eq. (17)

Thus, in the present study the following analysis is retained: Setting $\xi_R = 1$ corresponds to the upper physical limit of the reduced variable, the top of the admissible range. At this limit, the complicated nonlinear structure of the implicit equation collapses to a remarkably simple result: the relative conductivity cannot exceed the value $Q^* = \pi$. For $Q^* < \pi$, the implicit equation admits a real solution with $0 < \xi_R < 1$; as Q^* increases toward π, the corresponding solution ξ_R moves toward 1. At $Q^* = \pi$, the solution reaches the physically admissible relative normal flow depth value $\xi_R \approx 0.85245$, and beyond this value the equation no longer has a physically meaningful root in the interval [0, 1]. This result has a clear hydraulic interpretation: $Q^* = \pi$ represents a theoretical upper bound for the relative conductivity compatible with the geometry and the governing resistance law. It defines the limit of validity of the model and of the fixed-point formulation. Any attempt to impose a relative conductivity larger than π would require a value of ξ_R outside its physical range, and therefore falls outside the domain of the derived equation. In other words, the implicit equation itself encodes a natural hydraulic constraint, and the computation at $\xi_R = 1$ reveals that this constraint is precisely $Q^* \leq \pi$.

Therefore, the governing implicit Eq. (17) is transformed into the following final expression:

$$\xi_{\text{R}} = \frac{1}{2} - F\left(\xi_{\text{R}}; Q^*\right) \tag{19}$$

Although the Lagrange−Burmann inversion theorem is a powerful analytical tool for extracting series solutions from implicit relations of the form $z = \mathrm{G}(z)$, it cannot be effectively applied to the present governing equation, i.e., Eq. (17). The difficulty arises from the highly nonlinear structure of the transformed implicit relationship, which involves nested square roots, inverse trigonometric functions, and a cube-root dependence inside a composite argument. Such a structure violates the fundamental requirements for the uniform convergence of the Lagrange−Burmann series.

In theory, the theorem allows one to express the solution as a power series in the parameter Q^*. However, because the implicit function contains strong nonlinearities, particularly the terms involving

$$\cos^{-1}(1 - 2z)$$, and

$$\sqrt{z(1 - z)}$$

the radius of convergence of the series becomes extremely small. As a result, the Lagrange−Burmann expansion is valid only in a very narrow neighbourhood of $Q^* = 0$. When Q^* increases even moderately, the coefficients of the expansion grow rapidly, and the resulting series ceases to converge. This is a direct consequence of the analytic structure of the function: the presence of branch points at $z = 0$ and $z = 1$ restricts the complex-analytic domain in which the series remains valid.

Moreover, the physical variable ξ_{R} ranges between 0 and 1, and its dependence on Q^* is not analytic across the entire interval. The square-root and inverse-cosine terms introduce curvature and non-polynomial behaviour that cannot be captured by a Taylor series of finite or moderate length. Even when the series converges near $Q^* = 0$, it does so extremely slowly, making the analytical approximation useless for practical hydraulic calculations. Beyond very small relative conductivity, the truncated series diverges or oscillates, providing incorrect values of ξ_{R}.

For these reasons, the Lagrange−Burmann theorem is theoretically inapplicable and practically ineffective for the implicit governing equation. The method cannot handle the severity of the nonlinearities, the restricted analytic domain, nor the branching behaviour introduced by the trigonometric inversion. This makes it impossible to derive an accurate closed-form power-series solution valid over the full hydraulic domain.

Consequently, the only viable approaches are the fixed-point formulation, or the Aitkens Δ^2 accelerating method. It avoids all issues associated with series convergence, is straightforward to implement, and demonstrates excellent stability and accuracy for all admissible values of the relative conductivity Q^*. The inability of the Lagrange−Burmann method to produce a convergent series across the domain further reinforces the superiority of the fixed-point and Aitken methods for solving the governing implicit equation.

Eq. (19) constitutes the optimal formulation for applying the fixed-point method or the Aitken's iterative procedure. In this form, the unknown variable ξ_R appears explicitly on the left-hand side and only within the function F on the right-hand side. This structure ensures a clean and mathematically well-posed iterative process in which each iteration consists solely of evaluating elementary algebraic and trigonometric expressions. No derivatives, no nested inversions, and no implicit rearrangements are required.

In addition, expressing the equation in the following form:

$$z = \Phi\left(z;Q^*\right) \tag{20}$$

with

$$z = \xi_R \tag{21}$$

and

$$\Phi\left(z\right) = \frac{1}{2} - \frac{1}{2}\cos\left[F\left(\xi_R;\, Q^*\right)\right] \tag{22}$$

creates an explicit contraction mapping for all physically admissible values of Q^*. Because $\Phi(z)$ is continuous and bounded on [0, 1], and because its slope remains modest across the entire hydraulic domain, the following iteration:

$$z_{n+1} = \Phi\left(z_n\right) \tag{23}$$

is numerically stable and converges rapidly. This stability is a direct consequence of the algebraic structure obtained after the transformation of the first implicit equation: the cosine inversion has been cleanly separated, the nonlinear terms have been grouped coherently, and the entire expression has been reduced to a simple fixed-point mapping in a single scalar variable.

This explicit form is therefore the ideal platform for the fixed-point method or Aitken Δ^2 accelerating iterative procedure. It eliminates the need for derivative evaluation, as required in Newton-type schemes, avoids the algebraic complexity of the original implicit relation, and provides a computationally efficient and conceptually transparent approach to solving for ξ_R. As a result, the method is both robust and easy to implement, making it the most practical and reliable numerical strategy for handling the underlying hydraulic problem.

Extensive testing confirms that the fixed-point iteration and the Aitken accelerating iterative procedure converge reliably for all physically relevant values of Q^*. Even under strong geometric nonlinearity, the iteration demonstrates stable behaviour, and only a few steps are required to reach an acceptably accurate solution. This stability arises from the natural monotonicity and bounded behaviour of the transformed operator $\Phi(z)$, which provides a smooth numerical path toward convergence, independently of the magnitude of Q^*.

Iterative computation of the aspect ratio in the rough reference model

The values of z, i.e., the values of the aspect ratio ξ_R in the rough reference model, are derived from Eq. (17), while adopting the following iterative process:

$$z_1 = \frac{1}{2} - \frac{1}{2}\cos\left\{ \left(Q^*\right)^{2/3} \left[\cos^{-1}\left(1 - 2z_0\right)\right]^{1/3} + 2\left(1 - 2z_0\right)\sqrt{z_0\left(1 - z_0\right)} \right\} \tag{17a}$$

$$z_2 = \frac{1}{2} - \frac{1}{2}\cos\left\{ \left(Q^*\right)^{2/3} \left[\cos^{-1}\left(1 - 2z_1\right)\right]^{1/3} + 2\left(1 - 2z_1\right)\sqrt{z_1\left(1 - z_1\right)} \right\} \tag{17b}$$

$$z_3 = \frac{1}{2} - \frac{1}{2}\cos\left\{ \left(Q^*\right)^{2/3} \left[\cos^{-1}\left(1 - 2z_2\right)\right]^{1/3} + 2\left(1 - 2z_2\right)\sqrt{z_2\left(1 - z_2\right)} \right\} \tag{17c}$$

$$z_{k+1} = \frac{1}{2} - \frac{1}{2}\cos\left\{ \left(Q^*\right)^{2/3} \left[\cos^{-1}\left(1 - 2z_k\right)\right]^{1/3} + 2\left(1 - 2z_k\right)\sqrt{z_k\left(1 - z_k\right)} \right\} \tag{17d}$$

The iterative process stops when z_k and z_{k+1} are sufficiently close, depending on the user's requirements.

Fast alternative to compute the aspect ratio in the rough reference model

Because of this intrinsic nonlinearity of the governing Eq.(17d), the most reliable and universally applicable approach is to rely on a numerically generated reference table of exact solutions. These are reported in Table 1 in the appendix that lists Q^* from 0 to π in increments of 0.01, and corresponding exact reduced flow depth z(exact). This tabulation provides a direct and dependable means to determine z without running any iterative solver.

The detailed dataset of

$$z(\text{exact}) = \xi_{\text{R}}\left(Q^*\right)$$

has been generated by solving the governing Eq. (17) numerically with high accuracy, tolerance 10^{-12}, using the bisection method which is a fully robust bracketing technique that guarantees convergence for any admissible value of the relative conductivity Q^*. Table 1 in the appendix reports 316 pairs (Q^*, z exact).

If the user's computed value of Q^* appears explicitly in Table 1 in appendix, e.g., 0.27, 1.35, 2.99, …, then the corresponding $z = z$ (exact) $= \xi_{\text{R}}$ may be taken directly. No further computation is needed. If Q^* does not coincide with a tabulated entry, the required value of z must be obtained by simple linear interpolation, as follows:

In all cases, the user's Q^* computed value must necessarily satisfy the following inequalities:

$$Q_k^* < Q^* < Q_{k+1}^* \tag{24}$$

In other words,

$$Q_k^* \text{ and } Q_{k+1}^*$$

are the two values of the relative conductivity reported in Table 1 which border the calculated value of Q^*. One may write what follows:

$$z_k = z_{\text{exact}}\left(Q_k^*\right), \text{ and } z_{k+1} = z_{\text{exact}}\left(Q_{k+1}^*\right) \tag{25}$$

Because: (1) the step $\Delta Q^* = 0.01$ is extremely fine, and (2) the function $z(Q^*)$ is smooth and monotonic, then the interpolated solution is as follows:

$$z\left(Q^*\right) = z_k + \frac{Q^* - Q_k^*}{Q_{k+1}^* - Q_k^*}\left(z_{k+1} - z_k\right) \tag{26}$$

This linear interpolation introduces negligible numerical absolute error, typically far below 10^{-4}. Thus, the interpolated value is effectively as accurate as the exact numerical solution of the implicit equation.

Many advantages of the Table 1-based approach can be listed as follows: (1) Universality: works for every admissible Q^* without special-case handling or solver instability, (2) No iteration required: eliminates the complexity of choosing initial guesses or handling slow convergence, (3) High accuracy: with the fine sampling step of 0.01, linear interpolation yields results with engineering-grade accuracy (errors < 0.01%), (4) Simplicity for practitioners: engineers can directly read or interpolate the correct z, making the method suitable for hand calculations, spreadsheets, and software implementation.

As a numerical example, suppose the user computes the following relative conductivity:

$$Q^* = 1.234$$

This value does not appear in Table 1 in appendix, which is tabulated with a step $\Delta Q^* = 0.01$.

Now, locate the two neighbouring tabulated values, and look for the two closest values in the table such that satisfying the overmentioned inequalities expressed by Eq. (24).

Herein:

$$Q_k^* = 1.23 \text{ and } Q_{k+1}^* = 1.24$$

From Table 1 in appendix, one may derive the following:

$$Q_k^* = 1.23\,,\ z_k = z_{\text{exact}}\left(1.23\right) = 0.4305176650$$

$$Q_{k+1}^* = 1.24\,,\ z_{k+1} = z_{\text{exact}}\left(1.24\right) = 0.4326072059$$

Now, apply the linear interpolation given by Eq. (26). The following is written:

$$z\left(1.234\right) = 0.4305176650 + \frac{1.234 - 1.23}{1.24 - 1.23}\left(0.4326072059 - 0.4305176650\right)$$

Thus

$$z_{\text{interpolation}}\left(1.234\right) = \xi_{\text{R}} = 0.43135348$$

If we solve the implicit Eq. (17d) numerically at $Q^* = 1.234$, we obtain the following exact reference value:

$$z_k = z_{\text{exact}}\left(1.234\right) = 0.4313539552$$

Thus, the Table 1-based approach produces the following relative deviation in percent:

$$\frac{\Delta z}{z} = 100 \times \frac{\left| z_{\text{interpolation}} - z_{\text{exact}} \right|}{z_{\text{exact}}}$$

The calculation provides the following final result:

$$\frac{\Delta z}{z} = 0.00011016\% \approx 0.00011\%$$

Thus, this relative deviation (%) is completely negligible for any practical hydraulic purpose. Consequently, the user can safely obtain $z = \xi_R$ with extremely high accuracy, without ever having to solve the implicit equation themselves.

RMM-based steps for the sought aspect ratio computation

The purpose of this part of the study is to present the sequence of steps required to correctly determine the relative normal flow depth ξ, hence the sought normal flow depth, for the circular conduit under consideration. To this end, let us assume that the user has measured the following geometric and hydraulic parameters, in a given circular conduit:

The discharge Q = 0.066441704 m^3/s, the diameter D = 1 m, the geometric slope S_0 = 0.0001, the kinetic viscosity of the flowing liquid $\nu = 10^{-6}$ m^2/s, and the absolute roughness ε = 0.001 m.

1. Compute the relative conductivity Q^* using Eq. (9) as follows:

$$Q^* = \frac{Q}{\sqrt{2g S_0 D^5}} = \frac{0.066441704}{\sqrt{2 \times 9.81 \times 0.0001 \times 1^5}}$$

The final result is as follows:

$$Q^* = 1.5$$

2. According to Table 1 in appendix, the above calculated relative conductivity corresponds to following relative normal flow depth in the rough reference model:

$$z = \xi_R = 0.4857987451$$

3. Assume the discharge Q_R and the diameter D_R of the rough reference model equal to their homologs in the conduit under consideration. Thus, the following can be written:

$$Q_R = Q = 0.066441704 \text{ m}^3/s \text{, and } D_R = D = 1\text{m}$$

Thus, the wetted cross-section area in the rough reference model is as follows:

$$A_R = \frac{D^2}{4}\left[\cos^{-1}(1-2\xi_R) - 2(1-2\xi_R)\sqrt{\xi_R(1-\xi_R)}\right]$$

The calculation leads to the following result:

$$A_R = 0.37849974 \text{ m}^2$$

The wetted perimeter in the rough reference model is given as follows:

$$P_R = D\cos^{-1}\left(1 - 2\xi_R\right)$$

Thus, calculation shows the following:

$$P_R = 1.54239\,\text{m}$$

In addition, the Reynolds number characterizing the flow in the rough reference model is as follows:

$$R_R = \frac{4Q}{P_R\,\nu}$$

One may derive the following result:

$$R_R = 17,230,8.441$$

Furthermore, the hydraulic diameter in the rough reference model is given as follows:

$$D_{h,R} = \frac{4A_R}{P_R}$$

Calculation leads to the following result:

$$D_{h,R} = 0.98159282\,\text{m}$$

4. The rough model method states that any linear dimension "L" of a conduit or channel and the linear dimension "L_R" of its rough reference model are related by the following relationship, applicable to the entire domain of the turbulent flow:

$$\text{L} = \psi\,\text{L}_R$$

This is the fundamental RMM relationship.

The parameter ψ is defined as the dimensionless correction factor of linear dimension, less than unity, which is governed by the following relationship (Achour and Bedjaoui, 2006; Achour, 2007; Achour and Bedjaoui, 2012):

$$\psi = 1.35\left[-\log\left(\frac{\varepsilon / D_{h,R}}{4.75} + \frac{8.5}{R_{e,R}}\right)\right]^{-2/5}$$

Thus, inserting the value of each parameter yields the following:

$$\psi = 0.81066752$$

5. In this step of calculation, assign to the rough reference model the following new diameter, in accordance with the overmentioned fundamental RMM relationship:

$$D_R = \frac{D}{\psi}$$

The final result is as follows:

$$D_R = 1.23355133\,\mathrm{m}$$

6. Compute then the new relative conductivity as follows:

$$Q^* = \frac{Q}{\sqrt{2g S_0 (D/\psi)^5}}$$

Calculation provides the following final result:

$$Q^* = 0.88756096$$

This new value of the relative conductivity does not appear in the Table 1 in appendix. Thus, to derive the corresponding relative normal depth ξ_R, one may use the linear interpolation which has been described in one of the previous sections. The relationship reads as follows:

$$z(Q^*) = z_k + \frac{Q^* - Q_k^*}{Q_{k+1}^* - Q_k^*}\left(z_{k+1} - z_k\right)$$

Now, locate the two neighbouring tabulated values, and look for the two closest values in the table such that satisfying the inequalities expressed by Eq. (24). Herein, Table 1 in appendix contains values at the following:

$Q_k^* = 0.88$ and $Q_{k+1}^* = 0.89$

Thus, Table 1 reports the following *z* exact values:

$z_k = z_{\mathrm{exact}}(0.88) = \xi_R = 0.3541305168$, and

$$z_{k+1} = z_{\mathrm{exact}}(0.89) = \xi_R = 0.3564250136$$

Using the linear interpolation, one may write the following:

$$z\left(Q^*\right) = \xi_R = 0.3541305168 + \left\{\frac{0.88756096 - 0.88}{0.89 - 0.88} \times \left(0.3564250136 - 0.3541305168\right)\right\}$$

The final result is as follows, noting that the new value of ξ_R corresponds exactly to the sought relative normal flow depth ξ, so that writing the following:

$$\xi_R = \xi = \frac{y_n}{D} = 0.35586538$$

Thus, the sought normal flow depth is as follows:

$$y_n = = 0.35586538 \times 1 = 0.35586538\,\mathrm{m} \approx 0.356\,\mathrm{m}$$

7. This step aims to verify the reliability of the preceding method and calculation. To do so, the discharge given in the problem statement is compared to the discharge Q derived from the Chezy's equation, which is recalled as follows:

$$Q = C\,A\sqrt{R_h\,S_0}$$

Where C is the Chezy's coefficient, and R_h is the hydraulic radius expressed as follows:

$$R_h = \frac{A}{P}$$

The wetted area A is given as follows:

$$A = \frac{D^2}{4}\left[\cos^{-1}\left(1 - 2\xi\right) - 2\left(1 - 2\xi\right)\sqrt{\xi_R\left(1 - \xi\right)}\right]$$

with ξ = 0.35586538, previously computed. So, the final result is as follows:

$$A = 0.25058636\,\mathrm{m}^2$$

The wetted perimeter is given as follows:

$$P = D\cos^{-1}\left(1 - 2\xi\right)$$

Thus, calculation gives the following:

$$P = 1.27837747\,\mathrm{m}$$

Then, the hydraulic radius is as follows:

$$R_h = \frac{A}{P} = \frac{0.25058636}{1.27837747} = 0.19601907\,\mathrm{m}$$

The Chezy's C – RMM coefficient is expressed by the following relationship:

$$C = \frac{8\sqrt{2g}}{\psi^{5/2}} = \frac{8\times\sqrt{2\times 9.81}}{0.81066752^{5/2}} = 59.8869997\,\mathrm{m}^{0.5}/s$$

So, the discharge Q according to Chezy is as follows:

$$Q = 59.8869997\times 0.25058636\times\sqrt{0.19601907\times 0.0001}\ \mathrm{m}^3/s$$

The final result is as follows:

$$Q = 0.06644145\,\mathrm{m}^3/s$$

Thus, the deviation between the previous calculated discharge and the discharge given in the problem statement is written as follows:

$$\frac{\Delta Q}{Q} = 100\times\frac{\left|0.06644145 - 0.066441704\right|}{0.066441704} = 0.00037479\,\%$$

These findings demonstrate that the proposed method for computing the normal depth in circular conduits, using numerically generated reference table of exact solutions and the rough model method (RMM), is both robust and highly dependable. The procedure exhibits excellent numerical stability across the full range of hydraulic and geometric conditions considered, ensuring that practitioners can apply it with complete confidence. Its accuracy, combined with its conceptual simplicity, makes it particularly suitable for engineering design, operational analyses, and routine hydraulic assessments.

CONCLUSION

This study establishes a physically consistent and computationally reliable framework for determining the normal flow depth in circular conduits by integrating the Darcy-Weisbach formulation with the Rough Model Method (RMM). Unlike classical approaches based on Manning's or Chezy's equations, which implicitly rely on resistance coefficients that vary with the unknown flow depth, the RMM formulation avoids this fundamental contradiction by constructing a hydraulically equivalent rough reference circular conduit model. The resulting implicit equation for the reduced normal flow depth ξ_R in the rough reference model is fully dimensionless, self-contained, and independent of any empirical roughness parameter.

A comprehensive convergence analysis demonstrates that many traditional analytical tools, particularly Lagrange-Burmann series expansions, fail to provide practical solutions because the associated series converge only in an extremely narrow region near the relative conductivity $Q^* = 0$. By contrast, the governing relationship has been derived and arranged in a form that is exceptionally well adapted to numerical resolution. Its final structure is algebraically simple, dimensionally consistent, and fully self-contained, making it ideal for implementation through iterative schemes over the full admissible range $Q^* \in [0, \pi]$. In particular, the relationship is naturally compatible with the fixed-point method, which can be applied directly without requiring supplementary transformations or restrictive assumptions. Moreover, the formulation readily accommodates convergence-enhancement strategies such as Aitken's Δ^2 acceleration, ensuring rapid and stable convergence even in regions where the nonlinearity of the hydraulic relationship is pronounced. This deliberate structuring not only guarantees numerical robustness across the full admissible range of relative conductivity but also enables practitioners to obtain highly accurate solutions with minimal computational effort.

Nevertheless, due to the strong nonlinearity of the governing relationship, the most robust and universally applicable approach is the use of a high-resolution numerical reference table. The exact values of ξ_R tabulated for $Q^* \in [0, \pi]$, combined with straightforward linear interpolation, yield results of engineering-grade accuracy with negligible deviation. This eliminates the need for iterative computations and guarantees consistent performance across all hydraulic conditions.

The methodology developed in this work offers a powerful and practical solution for hydraulic engineers. It restores physical consistency to the normal-flow depth problem, ensures numerical robustness, and provides a simple yet highly accurate tool for real-world applications. The approach is therefore recommended as a reliable standard for determining normal flow depth in circular conduits and can be extended to other geometries within the RMM framework.

Declaration of competing interest

The author declares that he has no known competing financial interests or personal relationships that could influence the work reported in this article.

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Appendix

Table 1: Exact $z = \xi_R$ values with respect to the relative conductivity Q^*, according to Eq. (17)

Q^*	z(exact) = ξ_R
0.00	0.0000000000
0.01	0.0341991404
0.02	0.0485557100
0.03	0.0596512217
0.04	0.0690601612
0.05	0.0773922491
0.06	0.0849601187
0.07	0.0919494492
0.08	0.0984811959
0.09	0.1046393585
0.10	0.1104850845
0.11	0.1160645272
0.12	0.1214135363
0.13	0.1265606165
0.14	0.1315288756
0.15	0.1363373545
0.16	0.1410019639
0.17	0.1455361603
0.18	0.1499514456
0.19	0.1542577422
0.20	0.1584636815
0.21	0.1625768271
0.22	0.1666038514
0.23	0.1705506757
0.24	0.1744225835
0.25	0.1782243132
0.26	0.1819601332
0.27	0.1856339050
0.28	0.1892491354
0.29	0.1928090209
0.30	0.1963164843
0.31	0.1997742069
0.32	0.2031846552
0.33	0.2065501049
0.34	0.2098726602
0.35	0.2131542720
0.36	0.2163967528
0.37	0.2196017902

0.38	0.2227709587
0.39	0.2259057301
0.40	0.2290074826
0.41	0.2320775087
0.42	0.2351170231
0.43	0.2381271687
0.44	0.2411090224
0.45	0.2440636007
0.46	0.2469918641
0.47	0.2498947212
0.48	0.2527730331
0.49	0.2556276163
0.50	0.2584592462
0.51	0.2612686597
0.52	0.2640565583
0.53	0.2668236099
0.54	0.2695704516
0.55	0.2722976912
0.56	0.2750059093
0.57	0.2776956610
0.58	0.2803674775
0.59	0.2830218673
0.60	0.2856593180
0.61	0.2882802970
0.62	0.2908852531
0.63	0.2934746172
0.64	0.2960488036
0.65	0.2986082106
0.66	0.3011532218
0.67	0.3036842063
0.68	0.3062015201
0.69	0.3087055063
0.70	0.3111964958
0.71	0.3136748083
0.72	0.3161407524
0.73	0.3185946263
0.74	0.3210367185
0.75	0.3234673080
0.76	0.3258866648
0.77	0.3282950505
0.78	0.3306927183
0.79	0.3330799140
0.80	0.3354568756
0.81	0.3378238343
0.82	0.3401810142
0.83	0.3425286333
0.84	0.3448669030
0.85	0.3471960289
0.86	0.3495162111
0.87	0.3518276440

0.88	0.3541305168
0.89	0.3564250136
0.90	0.3587113139
0.91	0.3609895925
0.92	0.3632600196
0.93	0.3655227612
0.94	0.3677779793
0.95	0.3700258319
0.96	0.3722664731
0.97	0.3745000534
0.98	0.3767267200
0.99	0.3789466164
1.00	0.3811598829
1.01	0.3833666569
1.02	0.3855670725
1.03	0.3877612611
1.04	0.3899493511
1.05	0.3921314682
1.06	0.3943077357
1.07	0.3964782742
1.08	0.3986432019
1.09	0.4008026347
1.10	0.4029566861
1.11	0.4051054676
1.12	0.4072490884
1.13	0.4093876559
1.14	0.4115212753
1.15	0.4136500499
1.16	0.4157740814
1.17	0.4178934695
1.18	0.4200083123
1.19	0.4221187061
1.20	0.4242247458
1.21	0.4263265246
1.22	0.4284241343
1.23	0.4305176650
1.24	0.4326072059
1.25	0.4346928443
1.26	0.4367746666
1.27	0.4388527578
1.28	0.4409272015
1.29	0.4429980804
1.30	0.4450654760
1.31	0.4471294686
1.32	0.4491901374
1.33	0.4512475608
1.34	0.4533018160
1.35	0.4553529794
1.36	0.4574011262
1.37	0.4594463312

1.38	0.4614886678
1.39	0.4635282089
1.40	0.4655650266
1.41	0.4675991920
1.42	0.4696307758
1.43	0.4716598477
1.44	0.4736864768
1.45	0.4757107316
1.46	0.4777326800
1.47	0.4797523892
1.48	0.4817699258
1.49	0.4837853559
1.50	0.4857987451
1.51	0.4878101584
1.52	0.4898196604
1.53	0.4918273153
1.54	0.4938331866
1.55	0.4958373376
1.56	0.4978398312
1.57	0.4998407298
1.58	0.5018400956
1.59	0.5038379903
1.60	0.5058344754
1.61	0.5078296121
1.62	0.5098234613
1.63	0.5118160836
1.64	0.5138075395
1.65	0.5157978891
1.66	0.5177871925
1.67	0.5197755095
1.68	0.5217628998
1.69	0.5237494229
1.70	0.5257351382
1.71	0.5277201051
1.72	0.5297043827
1.73	0.5316880304
1.74	0.5336711071
1.75	0.5356536720
1.76	0.5376357842
1.77	0.5396175029
1.78	0.5415988872
1.79	0.5435799962
1.80	0.5455608892
1.81	0.5475416257
1.82	0.5495222650
1.83	0.5515028668
1.84	0.5534834908
1.85	0.5554641969
1.86	0.5574450453
1.87	0.5594260962

1.88	0.5614074101
1.89	0.5633890480
1.90	0.5653710707
1.91	0.5673535397
1.92	0.5693365167
1.93	0.5713200636
1.94	0.5733042428
1.95	0.5752891170
1.96	0.5772747494
1.97	0.5792612036
1.98	0.5812485436
1.99	0.5832368340
2.00	0.5852261396
2.01	0.5872165261
2.02	0.5892080595
2.03	0.5912008066
2.04	0.5931948346
2.05	0.5951902115
2.06	0.5971870059
2.07	0.5991852870
2.08	0.6011851249
2.09	0.6031865904
2.10	0.6051897551
2.11	0.6071946914
2.12	0.6092014726
2.13	0.6112101728
2.14	0.6132208673
2.15	0.6152336320
2.16	0.6172485441
2.17	0.6192656817
2.18	0.6212851241
2.19	0.6233069518
2.20	0.6253312463
2.21	0.6273580904
2.22	0.6293875682
2.23	0.6314197653
2.24	0.6334547683
2.25	0.6354926656
2.26	0.6375335470
2.27	0.6395775037
2.28	0.6416246288
2.29	0.6436750167
2.30	0.6457287639
2.31	0.6477859686
2.32	0.6498467308
2.33	0.6519111526
2.34	0.6539793380
2.35	0.6560513932
2.36	0.6581274266
2.37	0.6602075490

2.38	0.6622918733
2.39	0.6643805153
2.40	0.6664735931
2.41	0.6685712275
2.42	0.6706735423
2.43	0.6727806641
2.44	0.6748927226
2.45	0.6770098505
2.46	0.6791321843
2.47	0.6812598634
2.48	0.6833930312
2.49	0.6855318347
2.50	0.6876764250
2.51	0.6898269573
2.52	0.6919835910
2.53	0.6941464900
2.54	0.6963158232
2.55	0.6984917642
2.56	0.7006744918
2.57	0.7028641904
2.58	0.7050610499
2.59	0.7072652664
2.60	0.7094770422
2.61	0.7116965860
2.62	0.7139241138
2.63	0.7161598487
2.64	0.7184040214
2.65	0.7206568710
2.66	0.7229186448
2.67	0.7251895994
2.68	0.7274700007
2.69	0.7297601248
2.70	0.7320602584
2.71	0.7343706994
2.72	0.7366917577
2.73	0.7390237557
2.74	0.7413670294
2.75	0.7437219290
2.76	0.7460888199
2.77	0.7484680836
2.78	0.7508601189
2.79	0.7532653429
2.80	0.7556841924
2.81	0.7581171251
2.82	0.7605646215
2.83	0.7630271859
2.84	0.7655053489
2.85	0.7679996687
2.86	0.7705107338
2.87	0.7730391652

2.88	0.7755856189
2.89	0.7781507889
2.90	0.7807354107
2.91	0.7833402645
2.92	0.7859661794
2.93	0.7886140382
2.94	0.7912847818
2.95	0.7939794153
2.96	0.7966990146
2.97	0.7994447331
2.98	0.8022178103
2.99	0.8050195812
3.00	0.8078514871
3.01	0.8107150877
3.02	0.8136120759
3.03	0.8165442940
3.04	0.8195137534
3.05	0.8225226573
3.06	0.8255734275
3.07	0.8286687369
3.08	0.8318115470
3.09	0.8350051550
3.10	0.8382532492
3.11	0.8415599786
3.12	0.8449300377
3.13	0.8483687743
3.14	0.8518823245
3.141	0.8522380580
3.1415	0.8524162330